Advanced Integration Techniques

Advanced approaches for solving many complex integrals using special functions, some transformations and complex analysis approaches

Third Version

ZAID ALYAFEAI

YEMEN

mailto:alyafey22@gmail.com

Contents

بِسْمِ اللَه الرَحمَنِ الرَحيمِ

Acknowledgement

I want to offer my sincerest gratitude to all those who supported me during my journey to finish this book. Especially my parents, sisters and friends who supported the idea of this book. I also want to thank my Math teachers at King Fahd University because I wouldn't be able to learn the advanced without having knowledge of the elementary. I also want to extend my thanks to all my friends on the different math forums like MMF, MHB and stack exchange without them I wouldn't be learning any thing.

Reviewers

A special thank for **Mohammad Nather Shaaban** for reviewing some parts of the book.

What is new?

The new version is all about contour integration using the concepts from complex analysis. One might deviate from such approaches because of the heavy theory behind them but I tried to give a brief overview of the theory before delving deep into approaches.

The future work

I have a plan to add many other sections. Basically I'll try to focus on transformations like Mellin and fourier transforms. Also many other functions like the Jacobi theta function and q-series.

Introduction

This book is a summary of working on advanced integrations for around five years. It collects many examples that I gathered during that period. The approaches taken to solve the integrals aren't necessarily the only and best methods but they are offered for the sake of explaining the topic. Most of the content of this book I already wrote on mathhelpboards.com during the past three years but I thought that publishing it using a pdf would be easier to read and distribute. The motivation behind this book is to allow those who are interested in solving complicated integrals to be able to use the different methods to solve them efficiently. When I started learning about these techniques I would suffer to get enough information about all the required approaches so I tried to collect every thing in just one book. You are free to distribute this book and use any of the methods to solve the integrals or use the same techniques. The methods used are not necessarily new or ground-breaking but as I said they introduce the concept as easy as possible.

To follow this book you have to be know the basic integration techniques like integration by parts, by substitution and by partial fractions. I don't assume that the readers know any other stuff from any other topics or advanced courses from mathematics. Usually the details that require deep knowledge of analysis or advanced topics are left or just touched upon lightly to give the reader some hints but not going into details.

After reading this book you should be able to solve many advanced integrals that you might face in engineering courses. I hope you enjoy reading this book and if you have any suggestions, comments or correction I will be happy to recieve them through my email `mailto:alyafey22@gmail.com` or this email `mailto:alyafey_22@hotmail.com`. Also I am avilable as a staff member at `http://www.mathhelpboards.com` if you have some questions that I could reply to you directly using Latex.

1 Differentiation under the integral sign

This is one of the most commonly used techniques to solve a numerous number of questions.
Assume that we have the following function of two variables

$$\int_a^b f(x,y)\,dx$$

Then we can differentiate with respect to y provided that f is continuous and has a partial continuous derivative on a chosen interval

$$F'(y) = \int_a^b f_y(x,y)\,dx$$

Now using this in many problems is not that clear you have to think a lot to get the required answer because many integrals are usually in one variable so you need to introduce the second variable and assume it is a function of two variables.

1.1 Example

Assume we want to solve the following integral

$$\int_0^1 \frac{x^2-1}{\log(x)}\,dx$$

That seems very difficult to solve but using this technique we can solve it easily. The crux move is to decide where to put the second variable! So the problem with the integral is that we have a logarithm in the denominator which makes the problem so difficult to tackle! Remember that we can get a natural logarithm if we differentiate exponential functions i.e $F(a) = 2^a \Rightarrow\; F'(a) = \log(2)\cdot 2^a$
Applying this to our problem

$$F(a) = \int_0^1 \frac{x^a-1}{\log(x)}\,dx$$

Now we take the partial derivative with respect to a

$$F'(a) = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a-1}{\log(x)}\right)\,dx = \int_0^1 x^a\,dx = \frac{1}{a+1}$$

Integrate with respect to a

$$F(a) = \log(a+1) + C$$

To find the value of the constant put $a = 0$

$$F(0) = \log(1) + C \implies C = 0$$

This implies that

$$\int_0^1 \frac{x^a - 1}{\log(x)}\, dx = \log\left(a + 1\right)$$

By this powerful method we were not only able to solve the integral we also found a general formula for some a where the function is differentiable in the second variable.

To solve our original integral put $a = 2$

$$\int_0^1 \frac{x^2 - 1}{\log(x)}\, dx = \log\left(2 + 1\right) = \log(3)$$

1.2 Example

Find the following integral

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx$$

So where do we put the variable a here? that doesn't seem to be straight forward , how do we proceed ?

Let us try the following

$$F(a) = \int_0^{\frac{\pi}{2}} \frac{\arctan(a\tan(x))}{\tan(x)} dx$$

Now differentiate with respect to a

$$F'(a) = \int_0^{\frac{\pi}{2}} \frac{1}{1 + (a\tan(x))^2} dx$$

It can be proved that

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (a\tan(x))^2} dx = \frac{\pi}{2(1 + a)}$$

Now Integrate both sides

$$F(a) = \frac{\pi}{2}\log(1 + a) + C$$

Substitute $a = 0$ to find $C = 0$

$$\int_0^{\frac{\pi}{2}} \frac{\arctan(a\tan(x))}{\tan(x)}\,dx = \frac{\pi}{2}\log(1+a)$$

Put $a = 1$ in order to get our original integral

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan(x)}\,dx = \frac{\pi}{2}\log(2)$$

1.3 Example

$$\int_0^{\infty} \frac{\sin(x)}{x}\,dx$$

This problem can be solved by many ways , but here we will try to solve it by differentiation. So as I showed in the previous examples it is generally not easy to find the function to differentiate. Actually this step might require trial and error techniques until we get the desired result, so don't just give up if an approach doesn't work!

Let us try this one

$$F(a) = \int_0^{\infty} \frac{\sin(ax)}{x}\,dx$$

If we differentiated with respect to a we would get the following

$$F'(a) = \int_0^{\infty} \cos(ax)\,dx$$

But unfortunately this integral doesn't converge, so this is not the correct one. Actually, the previous theorem will not work here because the integral is improper.

So let us try the following

$$F(a) = \int_0^{\infty} \frac{\sin(x)e^{-ax}}{x}\,dx$$

Take the derivative

$$F'(a) = -\int_0^{\infty} \sin(x)e^{-ax}\,dx$$

Use integration by parts twice

$$F'(a) = -\int_0^{\infty} \sin(x)e^{-ax}\,dx = \frac{-1}{a^2+1}$$

Integrate both sides

$$F(a) = -\arctan(a) + C$$

To find the value of the constant take the limit as a grows large

$$C = \lim_{a\to\infty} F(a) + \arctan(a) = \frac{\pi}{2}$$

So we get our $F(a)$ as the following

$$F(a) = -\arctan(a) + \frac{\pi}{2}$$

For a = 0 we have

$$\int_0^\infty \frac{\sin(x)}{x} = \frac{\pi}{2}$$

2 Laplace Transform

2.1 Basic Introduction

Laplace transform is a powerful integral transform. It can be used in many applications. For example, it can be used to solve Differential Equations and its rules can be used to solve integration problems.
The basic definition of Laplace transform

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

This integral will converge when

$$\mathrm{Re}(s) > a \ , \ |f(t)| \leq M e^{at}$$

Let us see the Laplace transform for some functions

2.1.1 Example

Find the Laplace transform of the following functions

1. $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st}\, dt = \frac{1}{s}$$

2. For $f(t) = t^n$ where $n \geq 0$

   We can prove using integration by parts

$$F(s) = \int_0^\infty e^{-st} t^n\, dt = \frac{n!}{s^{n+1}}$$

3. For the geometric function $f(t) = \cos(at)$, Use integration by parts

$$F(s) = \int_0^\infty e^{-st} \cos(at)\, dt = \frac{s}{s^2 + a^2}$$

2.2 Example

Find the following integral

$$\int_0^\infty e^{-2t} t^3 \, dt$$

We can directly use the formula in the previous example

$$\int_0^\infty e^{-st} t^n \, dt = \frac{n!}{s^{n+1}}$$

Here we have $s = 2$ and $n = 3$

$$\int_0^\infty e^{-2t} t^3 \, dt = \frac{3!}{2^{3+1}} = \frac{3}{8}$$

2.3 Convolution

Define the following integral

$$(f * g)(t) = \int_0^t f(s) g(t - s) \, ds$$

Then we have the following

$$\mathcal{L}\left((f * g)(t)\right) = \mathcal{L}(f(t))\mathcal{L}(g(t))$$

2.4 Inverse Laplace transform

So, basically you are given $F(s)$ and we want to get $f(t)$ this is denoted by

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}^{-1}(F(s)) = f(t)$$

2.4.1 Example

Find the inverse Laplace transform of

1. $F(s) = \frac{1}{s^3}$

   We use the results applied previously

$$\mathcal{L}(t^2) = \frac{2!}{s^3} \Rightarrow \frac{1}{2}\mathcal{L}(t^2) = \frac{1}{s^3}$$

Now take the inverse to both sides

$$\frac{t^2}{2} = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right)$$

2. $F(s) = \frac{s}{s^2+4}$

   we can use the Laplace of cosine to deduce

$$\cos(2t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)$$

Exercises

Find the Laplace transform

$$\sin(at)$$

Find the inverse Laplace

$$\frac{1}{s^{n+1}}$$

2.5 Interesting results

2.5.1 Example

Prove the following

$$\beta(x,y) = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

β is the Beta function and Γ is the Gamma function. We will take enough time and examples to explain both functions in the next sections.

proof

We need convolution rule we described earlier

Let us choose some functions f and g

$$f(t) = t^x \ , \ g(t) = t^y$$

Hence we get

$$(t^x * t^y) = \int_0^t s^x(t-s)^y\,ds$$

So by the convolution rule we have the following

$$\mathcal{L}(t^x * t^y) = \mathcal{L}(t^x)\mathcal{L}(t^y)$$

We can now use the Laplace of the power

$$\mathcal{L}(t^x * t^y) = \frac{x! \cdot y!}{s^{x+y+2}}$$

Notice that we need to find the inverse of Laplace $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}(\mathcal{L}(t^x * t^y)) = \mathcal{L}^{-1}\left(\frac{x! \cdot y!}{s^{x+y+2}}\right) = t^{x+y+1}\frac{x! \cdot y!}{(x+y+1)!}$$

So we have the following

$$(t^x * t^y) = t^{x+y+1}\frac{x! \cdot y!}{(x+y+1)!}$$

By definition we have

$$t^{x+y+1}\frac{x! \cdot y!}{(x+y+1)!} = \int_0^t s^x(t-s)^y\,ds$$

Now put $t = 1$ we get

$$\frac{x! \cdot y!}{(x+y+1)!} = \int_0^1 s^x(1-s)^y\,ds$$

By using that $n! = \Gamma(n+1)$ we deduce that

$$\int_0^1 s^x(1-s)^y\,ds = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$$

which can be written as

$$\int_0^1 s^{x-1}(1-s)^{y-1}\,ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

2.5.2 Example

Prove the following

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty \mathcal{L}(f(t))\,ds$$

proof

we know from the definition

$$\int_0^\infty \mathcal{L}(f(t))\,ds = \int_0^\infty \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds$$

Now by the Fubini theorem we can rearrange the double integral

$$\int_0^\infty f(t)\left(\int_0^\infty e^{-st}\,ds\right) dt$$

The integral inside the parenthesis

$$\int_0^\infty e^{-st}\,ds = \frac{1}{t}$$

Now substitute this value in the integral

$$\int_0^\infty \frac{f(t)}{t}\,dt$$

2.5.3 Example

Find the following integral

$$\int_0^\infty \frac{\sin(t)}{t}\,dt$$

This is not the first time we see this integral and not the last . We have seen that we can find it using differentiation under the integral sign.

Let us use the previous example

$$\int_0^\infty \frac{\sin(t)}{t}\,dt = \int_0^\infty \mathcal{L}(\sin(t))\,ds$$

We can prove that

$$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$$

Substitute in our integral

$$\int_0^\infty \frac{ds}{1+s^2} = \tan^{-1}(s)|_{s=\infty} - \tan^{-1}(s)|_{s=0} = \frac{\pi}{2}$$

3 Gamma Function

The gamma function is used to solve many interesting integrals, here we try to define some basic properties, prove some of them and take some examples.

3.1 Definition

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x \, dt$$

For the first glance that just looks like the Laplace Transform, actually they are closely related.
So let us for simplicity assume that $x = n$ where $n \geq 0$ (is an integer)

$$\Gamma(n+1) = \int_0^\infty e^{-t}\, t^n \, dt$$

We can use the Laplace transform

$$\int_0^\infty e^{-t}\, t^n = \frac{n!}{s^{n+1}}|_{s=1} = n!$$

So we see that there is a relation between the gamma function and the factorial. We will assume for the time being that the gamma function is defined as the following

$$n! = \Gamma(n+1)$$

This definition is somehow limited but it will be soon replaced by a stronger one.

3.2 Example

Find the following integrals

$$\int_0^\infty e^{-t} t^4 \, dt$$

By definition this can be replaced by

$$\int_0^\infty e^{-t} t^4 \, dt = \Gamma(4+1) = 4! = 24$$

3.3 Example

Solving the following integrals

1.
$$\int_0^\infty e^{-t^2} t\,dt$$

We need a substitution before we go ahead, so let us start by putting $x = t^2$ so the integral becomes

$$\frac{1}{2}\int_0^\infty e^{-x}x^{\frac{1}{2}} \cdot x^{-\frac{1}{2}}\,dt = \frac{1}{2}\Gamma(1+0) = \frac{1}{2}$$

2.
$$\int_0^1 \log(t)\,t^2\,dt$$

we use the substitution $t = e^{-\frac{x}{2}}$

$$\int_0^1 \log(t)\,t^2\,dt = -\frac{1}{4}\int_0^\infty e^{-\frac{3x}{2}} \cdot x\,dx$$

Using another substitution $t = \frac{3x}{2}$

$$\frac{-1}{9}\int_0^\infty e^{-x}\,x\,dx = \frac{-\Gamma(2)}{9} = \frac{-1}{9}$$

It is an important thing to get used to the symbol Γ. I am sure that you are saying that this seems elementary, but my main aim here is to let you practice the new symbol and get used to solving some problems using it.

3.4 Exercises

Prove that

$$\frac{\Gamma(5)\cdot\Gamma(2)}{\Gamma(7)} = \frac{1}{30}$$

Find the following integral

$$\int_0^\infty e^{-\frac{1}{60}t}\,t^{20}\,dt$$

3.5 Extension

For simplicity we assumed that the gamma function only works for positive integers. This definition was so helpful as we assumed the relation between gamma and factorial. Actually, this restricts the gamma function, we want to exploit the real strength of this function. Hence, we must extend the gamma function to work for all real numbers except for some values. Actually we will see soon that we can extend it to work for all complex numbers except where the function has poles.

3.5.1 Theorem

Using the integral representation we can extend the gamma function to $x > -1$.

proof

We need only consider the case when $-1 < x < 0$.

Near infinity we have the following

$$\left| \int_0^\infty e^{-t}\, t^x \, dt \right| \leq \int_\epsilon^\infty e^{-t} dt < \infty$$

Near zero when $x = -z$ we have the following

$$\left| \int_0^\epsilon \frac{e^{-t}}{t^z} \, dt \right| \sim \int_0^\epsilon \frac{1}{t^z} dt < \infty$$

3.5.2 Reduction formula

$$\Gamma(x+1) = x\Gamma(x)$$

This can be proved through integration by parts for $x > 0$. Actually this representation allows us to extend the gamma function for all real numbers for non-negative integers. In terms of complex analysis this function is analytic except at non-positive integers where it has poles.

3.6 Other Representations

3.6.1 Euler Representation

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z}{z} \prod_{k=1}^{n} \frac{k}{k+z} = \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}}$$

proof

Note that

$$\Gamma(z+n+1) = \Gamma(z+1) \prod_{k=1}^{n} (k+z)$$

Which indicates that

$$\prod_{k=1}^{n}(k+z)=\frac{\Gamma(z+n+1)}{z\Gamma(z)}$$

Also note that

$$\prod_{k=1}^{n}k=n!$$

Hence we have

$$\lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{k+z}=\Gamma(z)\lim_{n\to\infty}\frac{n^z\times n!}{\Gamma(z+n+1)}$$

Hence we must show that

$$\lim_{n\to\infty}\frac{n^z\times n!}{\Gamma(z+n+1)}=1$$

Note that by Stirling formula

$$\Gamma(z+n+1)\sim\sqrt{2\pi}(n+z)^{n+z+1/2}e^{-(n+z)}$$

and

$$n!\sim\sqrt{2\pi}n^{n+1/2}e^{-n}$$

Hence we have by

$$\lim_{n\to\infty}\frac{n^z\times(\sqrt{2\pi}n^{n+1/2}e^{-n})}{\sqrt{2\pi}(n+z)^{n+z+1/2}e^{-(n+z)}}=\lim_{n\to\infty}\frac{n^n}{(n+z)^n e^{-z}}=1$$

Note that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=e^z$$

To prove the other product formula note that

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z=\frac{\prod_{k=1}^{n}(1+k)^z}{\prod_{k=1}^{n}k^z}=(n+1)^z\sim n^z$$

Hence we deduce

$$\lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{k+z}=\lim_{n\to\infty}\frac{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z}{z}\prod_{k=1}^{n}\frac{1}{1+\frac{z}{k}}=\frac{1}{z}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}}$$

3.6.2 Example

Prove that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \prod_{k=0}^{\infty}\left[\left(1+\frac{z}{x+k}\right)\left(1-\frac{z}{y+k}\right)\right]$$

proof

Start by

$$\Gamma(z) = \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{k+z}$$

We have

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \lim_{n\to\infty}\frac{\left(\frac{n^x}{x}\prod_{k=1}^{n}\frac{k}{k+x}\right)\left(\frac{n^y}{y}\prod_{k=1}^{n}\frac{k}{k+y}\right)}{\left(\frac{n^{x+z}}{x+z}\prod_{k=1}^{n}\frac{k}{k+x+z}\right)\left(\frac{n^{y-z}}{y-z}\prod_{k=1}^{n}\frac{k}{k+y-z}\right)}$$

By simplifications we have

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \lim_{n\to\infty}\frac{(x+z)(y-z)}{xy}\prod_{k=1}^{n}\frac{(k+x+z)(k+y-z)}{(k+x)(k+y)}$$

This simplifies to

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \prod_{k=0}^{\infty}\frac{(k+x+z)(k+y-z)}{(k+x)(k+y)} = \prod_{k=0}^{\infty}\left[\left(1+\frac{z}{x+k}\right)\left(1-\frac{z}{y+k}\right)\right]$$

3.6.3 Weierstrass Representation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}$$

where γ is the Euler constant

proof

Take logarithm to the Euler representation

$$\log z\Gamma(z) = \lim_{n\to\infty} z\sum_{k=1}^{n}\left(\log\left(1+k\right)-\log(k)\right)-\sum_{k=1}^{n}\log\left(1+\frac{z}{k}\right)$$

Note the alternating sum

$$\sum_{k=1}^{n}\left(\log\left(1+k\right)-\log(k)\right) = \log(n+1)$$

Hence we have

$$\log z\Gamma(z) = \lim_{n\to\infty} z\log(n+1) - \sum_{k=1}^{n} \log\left(1+\frac{z}{k}\right)$$

Now we can use the harmonic numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{n}$$

Add and subtract zH_{n+1}

$$\log z\Gamma(z) = \lim_{n\to\infty} z\log(n+1) - zH_{n+1} + \sum_{k=1}^{n}\left[\log\left(1+\frac{z}{k}\right)^{-1} + \frac{z}{k}\right] + \frac{z}{n+1}$$

The last term goes to zero and by definition we have the Euler constant

$$\gamma = \lim_{n\to\infty} H_n - \log(n)$$

Hence the first term is the Euler constant

$$\log z\Gamma(z) = -z\gamma + \sum_{k=1}^{\infty} \log\left(1+\frac{z}{k}\right)^{-1} + \frac{z}{k}$$

By taking the exponent of both sides

$$z\Gamma(z) = e^{-\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}}$$

3.7 Laurent expansion

$$\Gamma(z) = \frac{1}{z} - \gamma + \frac{1}{2!}(\gamma^2 + \zeta(2))z + O(z^2)$$

proof

Note that $f(z) = \Gamma(z+1)$ has a Maclurain expansion near 0

$$\Gamma(z+1) = \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} z^k$$

For the first term

$$f(0) = \Gamma(1+0) = 1$$

For the second term

$$\frac{f'(0)}{1!} = \Gamma'(1)$$

To find the derivative, note that by the Weierstrass representation

$$\log\Gamma(z) \ = -\gamma z - \log(z) + \sum_{n=1}^{\infty} \log\left(1+\frac{z}{n}\right)^{-1} + \frac{z}{n}$$

By taking the derivaive we have

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty}\frac{1}{k(z+k)}$$

Hence we have

$$\Gamma'(1) = -\gamma - 1 + \sum_{k=1}^{\infty}\frac{1}{k(1+k)} = -\gamma - 1 + 1 = -\gamma$$

For the third term

$$\frac{f''(0)}{2!} = \frac{\Gamma''(1)}{2}$$

Taking the second derivative

$$\frac{\Gamma''(z)\Gamma(z) - (\Gamma'(z))^2}{\Gamma^2(z)} = \frac{1}{z^2} + \sum_{k=1}^{\infty}\frac{1}{(z+k)^2}$$

Which indicates that

$$\Gamma''(1) = (\Gamma'(z))^2 + 1 + \sum_{k=1}^{\infty}\frac{1}{(1+k)^2} = \gamma^2 + \zeta(2)$$

Hence we deduce that

$$\Gamma(z+1) = 1 - \gamma z + \frac{1}{2!}(\gamma^2 + \zeta(2))z^2 + O(z^3)$$

Dividing by z we get our result.

3.8 Example

Find the integral

$$\int_0^{\infty}\frac{e^{-t}}{\sqrt{t}}\,dt$$

Now according to our definition this is equal to $\Gamma\left(\frac{1}{2}\right)$ but this value can be represented using elementary functions as follows

Let us first make a substitution $\sqrt{t} = x$

$$2\int_0^{\infty} e^{-x^2}\,dx$$

Now to find this integral we need to do a simple trick, start by the following

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\cdot\left(\int_0^\infty e^{-x^2}\,dx\right)$$

Since x is a dummy variable we can put

$$\left(\int_0^\infty e^{-x^2}\,dx\right)\cdot\left(\int_0^\infty e^{-y^2}\,dy\right)$$

Now since they are two independent variables we can do the following

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$

Now by polar substitution we get

$$\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

The inner integral is $\frac{1}{2}$, hence we get

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}\,d\theta = \frac{\pi}{4}$$

So we have

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

Take the square root to both sides

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$$

So we have our result

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

3.9 More values

We can use the reduction formula and the value of $\Gamma(1/2)$ to deduce other values. Assume that we want to find

$$\Gamma\left(\frac{3}{2}\right)$$

If we used this property we get

$$\Gamma\left(1+\frac{1}{2}\right)=\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2}$$

Not all the time the result will be reduced to a simpler form as the previous example. For example we don't know how to express $\Gamma(\frac{1}{4})$ in a simpler form but we can approximate its value

$$\Gamma\left(\frac{1}{4}\right)\approx 3.6256\cdots$$

Hence we just solve some integrals in terms of gamma function since we don't know a simpler form. For example solve the integral

$$\int_0^\infty e^{-t}t^{\frac{1}{4}}\,dt$$

we know by definition of gamma function that this reduces to

$$\int_0^\infty e^{-t}t^{\frac{1}{4}}\,dt=\Gamma\left(\frac{5}{4}\right)=\frac{\Gamma\left(\frac{1}{4}\right)}{4}$$

We have seen that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ but what about $\Gamma\left(\frac{-1}{2}\right)$?

By the reduction formula

$$\Gamma\left(1-\frac{1}{2}\right)=\frac{-1}{2}\Gamma\left(\frac{-1}{2}\right)$$

so we have that

$$\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$$

Then we can prove that any fraction where the denominator equals to 2 and the numerator is odd can be reduced into

$$\Gamma\left(\frac{2n+1}{2}\right)=C\,\Gamma\left(\frac{1}{2}\right)\ ,\ C\in\mathbb{Q}\ ,\ n\in\mathbb{Z}$$

3.10 Legendre Duplication Formula

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!}{4^n n!}\sqrt{\pi}$$

proof

For the proof we use induction by assuming $n \geq 0$. If $n = 0$ we have our basic identity

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

Now we need to prove that

$$\Gamma\left(\frac{1}{2}+n\right)=\frac{(2n)!}{4^n n!}\sqrt{\pi} \implies \Gamma\left(\frac{1}{2}+n+1\right)=\frac{(2n+2)!}{4^{n+1}(n+1)!}\sqrt{\pi}$$

Now we use the reduction formula

$$\Gamma\left(\frac{1}{2}+n+1\right)=\frac{1+2n}{2}\Gamma\left(\frac{1}{2}+n\right)$$

By the inductive step we have

$$\frac{1+2n}{2}\Gamma\left(\frac{1}{2}+n\right)=\frac{1+2n}{2}\cdot\frac{(2n)!}{4^n n!}\sqrt{\pi}$$

We can multiply and divide by $2n+2$

$$\frac{1+2n}{2}\cdot\frac{(2n)!}{4^n n!}\sqrt{\pi}\cdot\frac{2n+2}{2n+2}=\frac{(2n+2)!}{4^{n+1}(n+1)!}\sqrt{\pi}$$

3.11 Example

$$\int_0^\infty \frac{e^{-t}\cosh(a\sqrt{t})}{\sqrt{t}}dt$$

we have a hyperbolic function

We know that we can expand $\cosh$ using power series

$$\cosh(x)=\sum_{n=0}^\infty \frac{x^{2n}}{(2n)!}$$

Let $x = a\sqrt{t}$

$$\cosh(a\sqrt{t})=\sum_{n=0}^\infty \frac{a^{2n}\cdot t^n}{(2n)!}$$

Substituting back in the integral we have

$$\int_0^\infty e^{-t}\sum_{n=0}^\infty \frac{a^{2n}\cdot t^n}{(2n)!\sqrt{t}}dt$$

Now since the series is always positive we can swap the integral and the series

$$\sum_{n=0}^\infty \frac{a^{2n}}{(2n)!}\left[\int_0^\infty e^{-t}t^{n-\frac{1}{2}}\,dt\right]$$

Hence we have by using the gamma function

$$\sum_{n=0}^\infty \frac{a^{2n}\,\Gamma\left(\frac{1}{2}+n\right)}{(2n!)}$$

Using LDF (Legendre Duplication Formula) we get

$$\sum_{n=0}^\infty \frac{a^{2n}}{(2n!)}\left(\frac{(2n)!}{4^n n!}\sqrt{\pi}\right)$$

By further simplification

$$\sqrt{\pi}\,\sum_{n=0}^\infty \frac{a^{2n}}{4^n\,n!}$$

Now that looks familiar since we know that

$$\sum_{n=0}^\infty \frac{z^n}{n!} = e^z$$

Putting $z = \frac{a^2}{4}$ and multiplying by $\sqrt{\pi}$ we get

$$\sqrt{\pi}\sum_{n=0}^\infty \frac{\left(\frac{a^2}{4}\right)^n}{n!} = \sqrt{\pi}\,e^{\frac{a^2}{4}}$$

So we have finally that

$$\int_0^\infty \frac{e^{-t}\cosh(a\sqrt{t})}{\sqrt{t}}dt \;=\; \sqrt{\pi}e^{\frac{a^2}{4}}$$

3.12 Euler's Reflection Formula

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad ,\forall z \notin \mathbb{Z}$$

proof

We have to use the sine infinite product formula

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z}\prod_{n=1}^\infty \left(1-\frac{z^2}{n^2}\right)^{-1}$$

Now we start by noting that

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z)$$

Now using the Weierstrass formula we have

$$-z\Gamma(z)\Gamma(-z) = -z \cdot \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{z/n} \cdot \frac{e^{\gamma z}}{-z}\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)^{-1} e^{-z/n}$$

This simplifies to

$$\frac{1}{z}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)^{-1} = \frac{\pi}{\sin(\pi z)}$$

3.13 Example

Find the following

1. $$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

   The first example we can write

   $$\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

   Now by ERF (Euler reflection formula) we have the following

   $$\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}\,\pi$$

2. $$\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(\frac{1-i}{2}\right)$$

   Using the same idea for the second one

   $$\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(1-\frac{1+i}{2}\right)$$

   This expression simplifies to

   $$\frac{\pi}{\sin\left(\frac{\pi(1+i)}{2}\right)} = \frac{\pi}{\cos\left(\frac{i\pi}{2}\right)}$$

By geometry to hyperbolic conversions we get

$$\frac{\pi}{\cosh\left(\frac{\pi}{2}\right)} = \pi \operatorname{sech}\left(\frac{\pi}{2}\right)$$

3.14 Example

Find the integral

$$\int_a^{a+1} \log\Gamma(x)dx$$

Let the following

$$f(a) = \int_a^{a+1} \log\Gamma(x)dx$$

Differentiate both sides

$$f'(a) = \log\Gamma(1+a) - \log\Gamma(a) = \log(a)$$

Integrate both sides

$$f(a) = a\log(a) - a + C$$

Let $a \to 0$

We have

$$C = \int_0^1 \log\Gamma(x)\,dx$$

By the reflection formula

$$\int_0^1 \log\Gamma(x)\,dx = \int_0^1 \log\pi\,dx - \int_0^1 \log\sin(\pi x)\,dx - \int_0^1 \log\Gamma(1-x)\,dx$$

Which implies that

$$2\int_0^1 \log\Gamma(x)\,dx = \int_0^1 \log\pi\,dx - \int_0^1 \log\sin(\pi x)\,dx = \log(2\pi) - \int_0^1 \log|2\sin(\pi x)|\,dx$$

Note that this is the Clausen Integral

$$\int_0^1 \log|2\sin(\pi x)|\,dx = \frac{2}{\pi}\int_0^{2\pi} \log|2\sin(x/2)|\,dx = \frac{2}{\pi}\mathrm{cl}_2(2\pi) = 0$$

Hence we finalize by

$$\int_0^1 \log \Gamma(x)\, dx = \frac{1}{2}\log(2\pi)$$

4 Beta Function

4.1 Representations

4.1.1 First integral formula

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = B(x,y)$$

It is related to the gamma function through the identity

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

We have proved this identity earlier when we discussed convolution.

We shall realize the symmetry of beta function that is to say

$$\beta(x,y) = \beta(y,x)$$

Beta function has many other representations all can be deduced through substitutions

4.1.2 Second integral formula

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt$$

4.1.3 Geometric representation

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}(t)\sin^{2y-1}(t)\,dt$$

The proofs are left to the reader as practice.

4.2 Example

Prove the following

$$\int_0^\infty \frac{1}{z^2+1}\,dz = \frac{\pi}{2}$$

proof

Put $z = \sqrt{t}$

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{t+1}\,dt$$

We can use the second integral representation by finding the values of x and y

$$x - 1 = \frac{-1}{2} \;\Rightarrow\; x = \frac{1}{2}$$

$$x + y = 1 \;\Rightarrow\; y = \frac{1}{2}$$

Hence we have

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)}\,dt = \frac{B(\frac{1}{2},\frac{1}{2})}{2} = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{2} = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{2} = \frac{\pi}{2}$$

4.3 Example

$$\int_0^\infty \frac{1}{(z^2+1)^2}\,dz$$

Using the same substitution as the previous example we get

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^2}\,dt$$

Then we can find the values of x and y

$$x - 1 = \frac{-1}{2} \;\Rightarrow\; x = \frac{1}{2}$$

$$x + y = 2 \;\Rightarrow\; y = \frac{3}{2}$$

Then

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^2}\,dt = \frac{B(\frac{1}{2},\frac{3}{2})}{2} = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})}{2} = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{4} = \frac{\pi}{4}$$

4.4 Example

Find the generalization

$$\int_0^\infty \frac{1}{(x^2+1)^n}\,dx\,,\;\forall\; n > \frac{1}{2}$$

Using the same substitution again

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^n}\,dt$$

Then we can find the values of x and y

$$x - 1 = \frac{-1}{2} \;\Rightarrow\; x = \frac{1}{2}$$

$$x + y = n \;\Rightarrow\; y = n - \frac{1}{2}$$

Then

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^n}\,dt = \frac{\Gamma(\frac{1}{2})\cdot\Gamma\left(n-\frac{1}{2}\right)}{2\Gamma(n)}$$

Now by LDF

$$\Gamma\left(n - \frac{1}{2}\right) = \frac{(2n-2)!\sqrt{\pi}}{4^{n-1}\,(n-1)!} = \frac{\Gamma\,(2n-1)\,\sqrt{\pi}}{4^{n-1}\,\Gamma(n)}$$

Substituting in our integral we have the following

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^n}\,dt = \frac{2\pi\,\cdot\Gamma\,(2n-1)}{4^n\cdot\Gamma^2(n)}$$

$$\int_0^\infty \frac{1}{(x^2+1)^n}\,dx = \frac{\pi\,\cdot\Gamma\,(2n-1)}{2^{2n-1}\cdot\Gamma^2(n)}$$

It is easy to see that for $n \in \mathbb{Z}^+$ we get a π multiplied by some rational number.

4.5 Example

$$\int_0^1 \frac{z^n}{\sqrt{1-z}}\,dz = 2\cdot\frac{(2n)!!}{(2n+1)!!}$$

Where the double factorial !! is defined as the following

$$n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1 & ;\text{if } n \text{ is odd} \\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2 & ;\text{if } n \text{ is even} \\ 1 & ;\text{if } n = 0 \end{cases}$$

The integral in hand can be rewritten as

$$\int_0^1 z^n\cdot(1-z)^{-\frac{1}{2}}\,dz$$

We find the variables x and y

$$x - 1 = n \;\Rightarrow\; x = n + 1$$

$$y - 1 = \frac{-1}{2} \;\Rightarrow\; y = \frac{1}{2}$$

This can be written as

$$\int_0^1 z^n \cdot (1 - z)^{-\frac{1}{2}}\, dz = B(n + 1, \frac{1}{2})$$

By some simplifications

$$B\left(n + 1, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n + 1)}{\Gamma\left(n + \frac{3}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma(n + 1)}{(n + \frac{1}{2})\Gamma\left(n + \frac{1}{2}\right)}$$

Now you shall realize that we must use LDF

$$\frac{\sqrt{\pi}\Gamma(n + 1)}{(n + \frac{1}{2})\Gamma\left(n + \frac{1}{2}\right)} = \frac{2\sqrt{\pi}\, n!}{(2n + 1)\sqrt{\pi}\,\frac{(2n)!}{4^n\, n!}} = \frac{2 \cdot 2^{2n}(n!)^2}{(2n)!}$$

Now we should separate odd and even terms in the denominator

$$2 \cdot \frac{2^{2n}(n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1)^2}{(2n \cdot (2n - 2)\, \cdots\, 4 \cdot 2)((2n + 1) \cdot (2n - 1) \cdots 3 \cdot 1)}$$

We insert 2^{2n} into the square to obtain

$$2 \cdot \frac{(2n \cdot (2n - 2) \cdots 6 \cdot 4 \cdot 2)^2}{(2n \cdot (2n - 2) \cdots 4 \cdot 2)((2n + 1) \cdot (2n - 1) \cdots 3 \cdot 1)} = 2 \cdot \frac{(2n)!!}{(2n + 1)!!}$$

4.6 Example

Find the following integral

$$\int_{-\infty}^{\infty} \left(1 + \frac{x^2}{n - 1}\right)^{-\frac{n}{2}}\, dx$$

First we shall realize the evenness of the integral

$$2\int_0^{\infty} \left(1 + \frac{x^2}{n - 1}\right)^{-\frac{n}{2}}\, dx$$

Let $t = \frac{x^2}{n-1}$

$$\sqrt{n - 1}\int_0^{\infty} \frac{t^{-\frac{1}{2}}}{(1 + t)^{\frac{n}{2}}}\, dt$$

Now we see that our integral becomes so familiar

$$\sqrt{n-1}\,B\left(\frac{1}{2},\frac{n-1}{2}\right)=\frac{\sqrt{\pi(n-1)}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

4.7 Example

Find the following integral

$$\int_0^\infty \frac{x^{-p}}{x^3+1}\,dx$$

Let us do the substitution $x^3 = t$

$$\frac{1}{3}\int_0^\infty \frac{t^{-\frac{p+2}{3}}}{t+1}\,dt$$

Now we should find x, y

$$x=\frac{1-p}{3}$$

$$y+x=1 \;\Rightarrow\; y=1-\frac{1-p}{3}$$

so we have our beta representation of the integral

$$\frac{B\left(\frac{1-p}{3},\frac{1-p}{3}\right)}{3}=\frac{\Gamma\left(\frac{1-p}{3}\right)\Gamma\left(1-\frac{1-p}{3}\right)}{3}$$

Now we should use ERF

$$\frac{\Gamma\left(\frac{1-p}{3}\right)\Gamma\left(1-\frac{1-p}{3}\right)}{3}=\frac{\pi}{3\,\sin\left(\frac{\pi(1-p)}{3}\right)}=\frac{\pi}{3}\,\csc\left(\frac{\pi-\pi p}{3}\right)$$

4.8 Example

Now let us try to find

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin^3 z}\,dz$$

Rewrite as

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} z\,\cos^0 z\,dx$$

This is the Geometric representation

$$2x - 1 = \frac{3}{2} \;\Rightarrow\; x = \frac{5}{4}$$

$$2y - 1 = 0 \;\Rightarrow\; y = \frac{1}{2}$$

Then

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} z \, dz = \frac{B\left(\frac{5}{4}, \frac{1}{2}\right)}{2} = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{4}\right)}$$

4.9 Example

Find the following integral

$$\int_0^{\frac{\pi}{2}} (\sin z)^i \cdot (\cos z)^{-i} \, dz$$

This is the geometric representation

$$2x - 1 = i \;\Rightarrow\; x = \frac{1+i}{2}$$

$$2y - 1 = -i \;\Rightarrow\; y = \frac{1-i}{2}$$

Then

$$\frac{1}{2}\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(\frac{1-i}{2}\right)$$

Now we see that we have to use ERF

$$\frac{1}{2}\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(1 - \frac{1+i}{2}\right) = \frac{\pi}{2\sin\left(\frac{\pi(1+i)}{2}\right)} = \frac{\pi}{2}\text{sech}\,\left(\frac{\pi}{2}\right)$$

4.10 Exercise

Prove

$$\int_0^{\infty} \frac{x^{2m+1}}{(ax^2 + c)^n} dx = \frac{m!\,(n-m-2)!}{2(n-1)!\,a^{m+1}\,c^{n-m-1}}$$

5 Digamma function

5.1 Definition

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

We call digamma function the logarithmic derivative of the gamma function. Using this we can define the derivative of the gamma function.

$$\Gamma'(x) = \psi(x)\,\Gamma(x)$$

5.2 Example

Find the derivative of

$$f(x) = \frac{\Gamma(2x+1)}{\Gamma(x)}$$

We can use the differentiation rule for quotients

$$\frac{2\Gamma'(2x+1)\Gamma(x) - \Gamma'(x)\Gamma(2x+1)}{\Gamma^2(x)}$$

which can be rewritten as

$$\frac{2\Gamma(2x+1)\psi(2x+1)\Gamma(x) - \psi(x)\Gamma(x)\Gamma(2x+1)}{\Gamma^2(x)} = \frac{\Gamma(2x+1)}{\Gamma(x)}\left(2\psi(2x+1) - \psi(x)\right)$$

5.3 Difference formulas

5.3.1 First difference formula

$$\psi(1-x) - \psi(x) = \pi\cot(\pi x)$$

proof

We know by ERF that

$$\Gamma(x)\Gamma(1-x) = \pi\csc(\pi x)$$

Now differentiate both sides

$$\psi(x)\Gamma(x)\Gamma(1-x) - \psi(1-x)\Gamma(x)\Gamma(1-x) = -\pi^2 \csc(\pi x)\,\cot(\pi x)$$

Which can be simplified

$$\Gamma(x)\Gamma(1-x)\left(\psi(1-x) - \psi(x)\right) = \pi^2 \csc(\pi x)\,\cot(\pi x)$$

Further simplifications using ERF results in

$$\psi(1-x) - \psi(x) = \pi\cot(\pi x)$$

5.3.2 Second difference formula

$$\psi(1+x) - \psi(x) = \frac{1}{x}$$

proof

Let us start by the following

$$\frac{\Gamma(1+x)}{\Gamma(x)} = x$$

Now differentiate both sides

$$\frac{\Gamma(1+x)}{\Gamma(x)}\left(\psi(1+x) - \psi(x)\right) = 1$$

Which simplifies to

$$\psi(1+x) - \psi(x) = \frac{\Gamma(x)}{\Gamma(1+x)} = \frac{1}{x}$$

5.4 Example

Find the following integral

$$\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx$$

Consider the general case

$$\int_0^\infty \frac{x^a}{(1+x^2)^2}\,dx$$

Use the following substitution $x^2 = t$

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{a-1}{2}}}{(1+t)^2}\,dt$$

By the beta function this is equivalent to

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{a-1}{2}}}{(1+t)^2}\,dt = \frac{1}{2}B\left(\frac{a+1}{2}, 2-\frac{a+1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(2-\frac{a+1}{2}\right)$$

Differentiate with respect to a

$$F'(a) = \frac{1}{4}\int_0^\infty \frac{\log(t)\,t^{\frac{a-1}{2}}}{(1+t)^2}\,dt = \frac{1}{4}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(2-\frac{a+1}{2}\right)\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(2-\frac{a+1}{2}\right)\right]$$

Now put $a = 0$

$$\frac{1}{4}\int_0^\infty \frac{\log(t)\,t^{\frac{-1}{2}}}{(1+t)^2}\,dt = \frac{1}{4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\left[\psi\left(\frac{1}{2}\right)-\psi\left(\frac{3}{2}\right)\right]$$

Now we use our second difference formula

$$\psi\left(\frac{1}{2}\right)-\psi\left(\frac{3}{2}\right) = -\left(\psi\left(1+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\right)\right) = -2$$

Also by some gamma manipulation we have

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

The integral reduces to

$$\frac{1}{4}\int_0^\infty \frac{\log(t)\,t^{\frac{-1}{2}}}{(1+t)^2}\,dt = -\frac{\pi}{4}$$

Putting $x^2 = t$ we have our result

$$\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx = -\frac{\pi}{4}$$

5.5 Series Representation

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^\infty \frac{x}{n(n+x)}$$

proof

We start by taking the logarithm of the Weierstrass representation of the gamma function

$$\log\left(\Gamma(x)\right) = -\gamma\,x - \log(x) + \sum_{n=1}^\infty -\log\left(1+\frac{x}{n}\right)+\frac{x}{n}$$

Now we shall differentiate with respect to x

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{\frac{-1}{n}}{1 + \frac{x}{n}} + \frac{1}{n}$$

Further simplification will result in the following

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}$$

5.6 Some Values

Find the values of

1. $\psi(1)$

   $$\psi(1) = -\gamma - 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

   It should be easy to prove that

   $$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

   Hence we have

   $$\psi(1) = -\gamma$$

2. $\psi\left(\frac{1}{2}\right)$

   $$\psi\left(\frac{1}{2}\right) = -\gamma - 2 + \sum_{n=1}^{\infty} \frac{1}{n(2n+1)}$$

   We need to find

   $$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)}$$

   We can start by

   $$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$$

   So we can prove easily that

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - 2\log(2)$$

Hence

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log(2)$$

5.7 Example

Prove that

$$\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\,dx = \gamma$$

proof

Let $x = e^{-t}$

$$\int_0^\infty \frac{1}{e^t - 1} - \frac{e^{-t}}{t}\,dt$$

Let the following

$$F(s) = \int_0^\infty \frac{t^s}{e^t - 1} - t^{s-1}e^{-t}\,dt = \zeta(s+1)\Gamma(s+1) - \Gamma(s)$$

Hence the limit

$$\lim_{s\to 0} \Gamma(s+1)\left(\zeta(s+1) - \frac{1}{s}\right) = \lim_{s\to 0} \zeta(s+1) - \frac{1}{s}$$

Use the expansion of the zeta function

$$\zeta(s+1) = \frac{1}{s} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!}(-s)^n$$

Hence the limit is equal to $\gamma_0 = \gamma$.

5.8 Integral representations

5.8.1 First Integral representation

$$\psi(a) = \int_0^\infty \frac{e^{-z} - (1+z)^{-a}}{z}\,dz$$

We begin with the double integral

$$\int_0^\infty \int_1^t e^{-xz}\,dx\,dz = \int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\,dz$$

Using fubini theorem we also have

$$\int_1^t \int_0^\infty e^{-xz}\,dz\,dx = \int_1^t \frac{1}{x}\,dx = \log t$$

Hence we have the following

$$\int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\,dz = \log(t)$$

We also know that

$$\Gamma'(a) = \int_0^\infty t^{a-1}e^{-t}\,\log t\,dt$$

Hence we have

$$\Gamma'(a) = \int_0^\infty t^{a-1}e^{-t}\left(\int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\,dz\right)dt = \int_0^\infty \int_0^\infty \frac{t^{a-1}e^{-t}e^{-z} - t^{a-1}e^{-t(z+1)}}{z}\,dz\,dt$$

Now we can use the fubini theorem

$$\Gamma'(a) = \int_0^\infty \int_0^\infty \frac{t^{a-1}e^{-t}e^{-z} - t^{a-1}e^{-t(z+1)}}{z}\,dt\,dz$$

$$\Gamma'(a) = \int_0^\infty \frac{1}{z}\left(e^{-z}\int_0^\infty t^{a-1}e^{-t}\,dt - \int_0^\infty t^{a-1}e^{-t(z+1)}\,dt\right)dz$$

But we can easily deduce using Laplace that

$$\int_0^\infty t^{a-1}e^{-t(z+1)}\,dt = \Gamma(a)\,(z+1)^{-a}$$

Aslo we have

$$\int_0^\infty t^{a-1}e^{-t}\,dt = \Gamma(a)$$

Hence we can simplify our integral to the following

$$\Gamma'(a) = \Gamma(a)\int_0^\infty \frac{e^{-z} - (1+z)^{-a}}{z}dz$$

$$\frac{\Gamma'(a)}{\Gamma(a)} = \psi(a) = \int_0^\infty \frac{e^{-z} - (1+z)^{-a}}{z}\,dz$$

5.8.2 Second Integral representation

$$\psi(s+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x}\,dx$$

proof

This can be done by noting that

$$\psi(s+1) = -\gamma + \sum_{n=1}^{\infty} \frac{s}{n(n+s)}$$

It is left as an exercise to prove that

$$\sum_{n=1}^{\infty} \frac{s}{n(n+s)} = \int_0^1 \frac{1-x^s}{1-x}\,dx$$

5.8.3 Third Integral representation

$$\psi(a) = \int_0^{\infty} \frac{e^{-t}}{t} - \frac{e^{-(at)}}{1-e^{-t}}\,dt$$

proof

Let $e^{-t} = x$

$$\int_0^1 -\frac{1}{\log(x)} - \frac{x^{a-1}}{1-x}\,dx$$

By adding and subtracting 1

$$-\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\,dx + \int_0^1 \frac{1-x^{a-1}}{1-x}\,dx$$

Using the second integral representation

$$-\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\,dx + \gamma + \psi(a)$$

We have already proved that

$$\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\,dx = \gamma$$

Finally we get

$$\int_0^{\infty} \frac{e^{-t}}{t} - \frac{e^{-(at)}}{1-e^{-t}}\,dt = -\gamma + \gamma + \psi(a) = \psi(a)$$

5.8.4 Fourth Integral representation

Prove that

$$\psi(z) = \log(z) - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi}-1)}dt \ \ ; \ \mathrm{Re}z > 0$$

We prove that

$$2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi}-1)}dt = \log(z) - \frac{1}{2z} - \psi(z)$$

First note that

$$\frac{2}{e^{2\pi t}-1} = \coth(\pi t) - 1$$

Also note that

$$\coth(\pi t) = \frac{1}{\pi t} + \frac{2t}{\pi}\sum_{k=1}^\infty \frac{1}{k^2+t^2}$$

Hence we conclude that

$$\frac{2t}{e^{2\pi t}-1} = \frac{1}{\pi} - t + \frac{2t^2}{\pi}\sum_{k=1}^\infty \frac{1}{k^2+t^2}$$

Substitute the value in the integral

$$\int_0^\infty \frac{1}{t^2+z^2}\left\{\frac{1}{\pi} - t + \frac{2t^2}{\pi}\sum_{k=1}^\infty \frac{1}{k^2+t^2}\right\}dt$$

The first integral

$$\frac{1}{\pi}\int_0^\infty \frac{1}{t^2+z^2}dt = \frac{1}{2z}$$

Since the second integral is divergent we put

$$\int_0^N \frac{t}{t^2+z^2}\,dt = \frac{1}{2}\log(N^2+z^2) - \log(z)$$

Also for the series

$$\frac{2}{\pi}\sum_{k=1}^N \int_0^\infty \frac{t^2}{(t^2+z^2)(t^2+k^2)}dt = \sum_{k=1}^N \frac{1}{k+z}$$

Which simplifies to

$$\sum_{k=1}^{N}\frac{1}{k+z}=\sum_{k=1}^{N}\frac{1}{k}-\sum_{k=1}^{N}\frac{z}{k(k+z)}=H_N-\sum_{k=1}^{N}\frac{z}{k(k+z)}$$

Now take the limit

$$\lim_{N\to\infty}-\frac{1}{2}\log(N^2+z^2)+\log(z)+H_N-\sum_{k=1}^{N}\frac{z}{k(k+z)}$$

Or

$$\lim_{N\to\infty}H_N-\log(N)+\log(z)-\sum_{k=1}^{\infty}\frac{z}{k(k+z)}$$

This simplifies to

$$\log(z)+\gamma-\sum_{k=1}^{\infty}\frac{z}{k(k+z)}=\log(z)-\frac{1}{z}-\psi(z)$$

Collecting the results we have

$$2\int_0^\infty\frac{t}{(t^2+z^2)(e^{2\pi}-1)}dt=\log(z)+\frac{1}{2z}-\frac{1}{z}-\psi(z)=\log(z)-\frac{1}{2z}-\psi(z)$$

5.9 Gauss Digamma theorem

Let p/q be a rational number with $0<p<q$ then

$$\psi\left(\frac{p}{q}\right)=-\gamma-\log(2q)-\frac{\pi}{2}\cot\left(\frac{p}{q}\pi\right)+2\sum_{k=1}^{q/2-1}\cos\left(\frac{2\pi pk}{q}\right)\log\left[\sin\left(\frac{\pi k}{q}\right)\right]$$

proof

The proof is omitted.

5.10 More results

Assume that $p=1$ and $q>1$ is an integer then

$$\psi\left(\frac{1}{q}\right)=-\gamma-\log(2q)-\frac{\pi}{2}\cot\left(\frac{\pi}{q}\right)+2\sum_{k=1}^{q/2-1}\cos\left(\frac{2\pi k}{q}\right)\log\left[\sin\left(\frac{\pi k}{q}\right)\right]$$

So for example

$$\psi\left(\frac{1}{3}\right) = \frac{1}{6}(-6\gamma - \pi\sqrt{3} - 9\log(3))$$
$$\psi\left(\frac{1}{4}\right) = \frac{1}{2}(-2\gamma - \pi - 6\log(2))$$
$$\psi\left(\frac{1}{6}\right) = -\gamma - \frac{1}{2}\sqrt{3}\pi - 2\log(2) - \frac{3}{2}\log(3)$$

5.11 Example

Find the following integral

$$\int_0^\infty e^{-at}\,\log(t)\,dt$$

We start by considering

$$F(b) = \int_0^\infty e^{-at}\,t^b\,dt$$

Now use the substitution $x = at$ we get

$$F(b) = \frac{1}{a}\int_0^\infty e^{-x}\left(\frac{x}{a}\right)^b dx$$

We can use the gamma function

$$F(b) = \frac{1}{a}\int_0^\infty e^{-x}\left(\frac{x}{a}\right)^b dx = \frac{\Gamma(b+1)}{a^{b+1}}$$

Now differentiate with respect to b

$$F'(b) = \frac{1}{a}\int_0^\infty e^{-x}\,\log\left(\frac{x}{a}\right)\left(\frac{x}{a}\right)^b dx = \frac{\Gamma(b+1)\psi(b+1)}{a^{b+1}} - \frac{\log(a)\Gamma(b+1)}{a^{b+1}}$$

Now put $b = 0$ and $at = x$

$$\int_0^\infty e^{-at}\,\log(t)\,dt = \frac{\psi(1) - \log(a)}{a} = -\frac{\gamma + \log(a)}{a}$$

5.12 Example

Prove the following

$$\int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\log x)}dx = \log\left\{\frac{\Gamma(b+c+1)\Gamma(c+a+1)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(a+b+c+1)}\right\}$$

proof

First note that since there is a log in the denominator that gives as an idea to use differentiation under the integral sign.

Let

$$F(c) = \int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\log x)} dx$$

Differentiate with respect to c

$$F'(c) = \int_0^1 \frac{(1-x^a)(1-x^b)x^c}{(1-x)} dx$$

By expanding

$$F'(c) = \int_0^1 \frac{(1-x^a-x^b+x^{a+b})x^c}{(1-x)} dx = \int_0^1 \frac{x^c - x^{a+c} - x^{b+c} + x^{a+b+c}}{(1-x)} dx$$

We can add and subtract one to use the second integral representation

$$F'(c) = \int_0^1 \frac{(x^c - 1) + (1 - x^{a+c}) + (1 - x^{b+c}) + (x^{a+b+c} - 1)}{(1-x)} dx$$

Distribute the integral over the terms

$$F'(c) = -\int_0^1 \frac{1-x^c}{1-x} dx + \int_0^1 \frac{1-x^{a+c}}{1-x} dx + \int_0^1 \frac{1-x^{b+c}}{1-x} dx - \int_0^1 \frac{1-x^{a+b+c}}{1-x} dx$$

Which simplifies to

$$F'(c) = -\psi(c+1) + \psi(a+c+1) + \psi(b+c+1) - \psi(a+b+c+1)$$

Integrate with respect to c

$$F(c) = -\log[\Gamma(c+1)] + \log[\Gamma(a+c+1)] + \log[\Gamma(b+c+1)] - \log[\Gamma(a+b+c+1)] + e$$

Which reduces to

$$\log\left[\frac{\Gamma(a+c+1)\Gamma(b+c+1)}{\Gamma(c+1)\Gamma(a+b+c+1)}\right] + e$$

Now put $c = 0$ we have

$$0 = \log\left[\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\right] + e$$

The constant

$$e = -\log\left[\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\right]$$

So we have the following

$$F(c) = \log\left[\frac{\Gamma(a+c+1)\Gamma(b+c+1)}{\Gamma(c+1)\Gamma(a+b+c+1)}\right] - \log\left[\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\right]$$

Hence we have the result

$$\int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\log x)}dx = \log\left\{\frac{\Gamma(b+c+1)\Gamma(c+a+1)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(a+b+c+1)}\right\}$$

5.13 Example

Find the following integral

$$\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax}\right)\frac{dx}{x}$$

Let us first use the substitution $t = ax$

$$\int_0^\infty \left(e^{-\frac{bt}{a}} - \frac{1}{1+t}\right)\frac{dt}{t}$$

Add and subtract e^{-t}

$$\int_0^\infty \left(e^{-t} - e^{-t} + e^{-\frac{bt}{a}} - \frac{1}{1+t}\right)\frac{dt}{t}$$

Separate into two integrals

$$\int_0^\infty \left(e^{-t} - \frac{1}{1+t}\right)\frac{dt}{t} + \int_0^\infty \frac{e^{-\frac{bt}{a}} - e^{-t}}{t}\,dt$$

The first integral is a representation of the Euler constant when $a = 1$

$$\int_0^\infty \left(e^{-t} - \frac{1}{1+t}\right)\frac{dt}{t} = -\gamma$$

We also proved

$$\int_0^\infty \frac{e^{-\frac{bt}{a}} - e^{-t}}{t}\,dt = -\log\left(\frac{b}{a}\right) = \log\left(\frac{a}{b}\right)$$

Hence the result

$$\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax}\right)\frac{dx}{x} = \log\left(\frac{a}{b}\right) - \gamma$$

5.14 Example

Find the following integral

$$\int_0^\infty e^{-ax}\left(\frac{1}{x}-\coth(x)\right)dx$$

By using the exponential representation of the hyperbolic functions

$$\int_0^\infty e^{-ax}\left(\frac{1}{x}-\frac{1+e^{-2x}}{1-e^{-2x}}\right)dx$$

Now let $2x=t$ so we have

$$\int_0^\infty e^{-\left(\frac{at}{2}\right)}\left(\frac{1}{t}-\frac{1+e^{-t}}{2(1-e^{-t})}\right)dt$$

$$\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}}{t}-\frac{e^{-\left(\frac{at}{2}\right)}+e^{\left(-\frac{at}{2}-t\right)}}{2(1-e^{-t})}\,dt$$

By adding and subtracting some terms

$$\int_0^\infty \frac{e^{-t}+e^{-\left(\frac{at}{2}\right)}-e^{-t}}{t}-\frac{e^{-\left(\frac{at}{2}\right)}+e^{-\left(\frac{at}{2}\right)}-e^{-\frac{at}{2}}+e^{-\left(\frac{at}{2}\right)-t}}{2(1-e^{-t})}\,dt$$

Separate the integrals

$$\int_0^\infty \frac{e^{-t}}{t}-\frac{e^{-\left(\frac{at}{2}\right)}}{1-e^{-t}}\,dt+\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}-e^{\left(-\frac{at}{2}-t\right)}}{2(1-e^{-t})}\,dt+\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}-e^{-t}}{t}\,dt$$

By using the third integral representation

$$\int_0^\infty \frac{e^{-t}}{t}-\frac{e^{-\left(\frac{at}{2}\right)}}{1-e^{-t}}\,dt=\psi\left(\frac{a}{2}\right)$$

The second integral reduces to

$$\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}-e^{-\left(\frac{at}{2}-t\right)}}{2(1-e^{-t})}=\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}}{2}\,dt=\frac{1}{a}$$

The third integral

$$\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}-e^{-t}}{t}\,dt=-\log\left(\frac{a}{2}\right)$$

By collecting the results

$$\int_0^\infty e^{-ax}\left(\frac{1}{x}-\coth(x)\right)dx=\psi\left(\frac{a}{2}\right)-\log\left(\frac{a}{2}\right)+\frac{1}{a}$$

5.15 Example

Prove that

$$\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(ax)} = \frac{1}{2}\psi\left(\frac{1}{2}+\frac{a}{2\pi}\right) - \frac{1}{2}\psi\left(\frac{a}{2\pi}\right) - \frac{2}{\pi}a$$

proof

$$\begin{aligned}
\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(ax)} &= \int_0^\infty \frac{x}{x^2+a^2}\frac{dx}{\sinh(x)} \\
&= \int_0^\infty \int_0^\infty e^{-at}\frac{\sin(xt)}{\sinh(x)}\,dt\,dx \\
&= \int_0^\infty e^{-at}\int_0^\infty \frac{\sin(xt)}{\sinh(x)}\,dx\,dt \\
&= \frac{\pi}{2}\int_0^\infty e^{-at}\tanh\left(\frac{\pi}{2}t\right)dt \\
&= \int_0^\infty e^{-zx}\tanh(x)\,dt \quad ; z = \frac{2}{\pi}a \\
&= \int_0^\infty \frac{e^{-zx}(1-e^{-2x})}{e^{-2x}+1}\,dx
\end{aligned}$$

By splitting the integral we have

$$\begin{aligned}
\int_0^\infty \frac{e^{-zx}}{e^{-2x}+1}\,dx &= \sum_{n\geq 0}\int_0^\infty e^{-x(2n+z)}\,dx \\
&= \sum_{n\geq 0}\frac{(-1)^n}{2n+z} \\
&= \frac{1}{4}\left(\psi\left(\frac{1}{2}+\frac{z}{4}\right) - \psi\left(\frac{z}{4}\right)\right)
\end{aligned}$$

$$\begin{aligned}
-\int_0^\infty \frac{-e^{-x(z+2)}}{e^{-2x}+1}\,dx &= -\sum_{n\geq 0}\frac{(-1)^n}{z+2+2n} \\
&= -\frac{1}{4}\left(-\psi\left(\frac{1}{2}+\frac{z}{4}\right) + \psi\left(1+\frac{z}{4}\right)\right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(ax)} &= \frac{1}{4}\left(2\psi\left(\frac{1}{2}+\frac{z}{4}\right) - \psi\left(1+\frac{z}{4}\right) - \psi\left(\frac{z}{4}\right)\right) \\
&= \frac{1}{2}\psi\left(\frac{1}{2}+\frac{z}{4}\right) - \frac{1}{2}\psi\left(\frac{z}{4}\right) - z \\
&= \frac{1}{2}\psi\left(\frac{1}{2}+\frac{a}{2\pi}\right) - \frac{1}{2}\psi\left(\frac{a}{2\pi}\right) - \frac{2}{\pi}a
\end{aligned}$$

Let $a = \pi/2$

$$\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(\frac{\pi}{2}x)} = \frac{1}{2}\psi\left(\frac{1}{2}+\frac{1}{4}\right) - \frac{1}{2}\psi\left(\frac{1}{4}\right) - 1$$
$$= \frac{\pi}{2}\cot(\pi/4) - 1$$
$$= \frac{\pi}{2} - 1$$

6 Zeta function

Zeta function is one of the most important mathematical functions. The study of zeta function isn't exclusive to analysis. It also extends to number theory and the most celebrating theorem of Riemann.

6.1 Definition

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

6.2 Bernoulli numbers

We define the Bernoulli numbers B_k as

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

Now let us derive some values for the Bernoulli numbers , rewrite the power series as

$$x = (e^x - 1) \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

By expansion

$$x = \left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \cdot \left(B_0 + B_1\, x + \frac{B_2}{2!}\, x^2 + \frac{B_3}{3!} x^3 + \cdots\right)$$

By multiplying we get

$$x = B_0 x + \left(B_1 + \frac{B_0}{2!}\right) x^2 + \left(\frac{B_0}{3!} + \frac{B_2}{2!} + \frac{B_1}{2!}\right) x^3 + \left(\frac{B_0}{4!} + \frac{B_1}{3!} + \frac{B_2}{2!\,2!} + \frac{B_3}{3!}\right) x^4 + \cdots$$

By comparing the terms we get the following values

$$B_0 = 1\,,\; B_1 = -\frac{1}{2} \;+\; B_2 = \frac{1}{6}\,,\; B_3 = 0\,,\; B_4 = -\frac{1}{30}, \cdots$$

Actually we also deduce that

$$B_{2k+1} = 0 \quad , \;\forall \quad k \in \mathbb{Z}^+$$

6.3 Relation between zeta and Bernoulli numbers

According to Euler we have the following relation

$$\zeta(2k) = (-1)^{k-1} B_{2k} \frac{2^{2k-1}}{(2k)!} \pi^{2k}$$

proof

We start by the product formula of the sine function

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

Take the logarithm to both sides

$$\log(\sin(z)) - \log(z) = \sum_{n=1}^{\infty} \log\left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

By differentiation with respect to z

$$\cot(z) - \frac{1}{z} = -2 \sum_{n=1}^{\infty} \frac{\frac{z}{n^2 \pi^2}}{1 - \frac{z^2}{n^2 \pi^2}}$$

By simple algebraical manipulation we have

$$z \cot(z) = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2 \pi^2} \left(\frac{1}{1 - \frac{z^2}{\pi^2 n^2}} \right)$$

Now using the power series expansion

$$\frac{1}{1 - \frac{z^2}{\pi^2 n^2}} = \sum_{k=0}^{\infty} \frac{1}{n^{2k} \pi^{2k}} z^{2k} \quad , |z| < \pi n$$

$$\frac{z^2}{n^2 \pi^2} \left(\frac{1}{1 - \frac{z^2}{\pi^2 n^2}} \right) = \sum_{k=0}^{\infty} \frac{1}{n^{2k+2} \pi^{2k+2}} z^{2k+2} = \sum_{k=1}^{\infty} \frac{1}{n^{2k} \pi^{2k}} z^{2k}$$

So the sums becomes

$$z \cot(z) = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^{2k}} \frac{z^{2k}}{\pi^{2k}}$$

Now if we invert the order of summation we have

$$z \cot(z) = 1 - 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \frac{z^{2k}}{\pi^{2k}} = 1 - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k}$$

Euler didn't stop here, he used power series for $z \cot(z)$ using the Bernoulli numbers.

Start by the equation

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

By putting $x = 2iz$ we have

$$\frac{2iz}{e^{2iz} - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} (2iz)^k$$

Which can be reduced directly to the following by noticing that $B_{2k+1} = 0$

$$z\cot(z) = 1 - \sum_{k=1}^{\infty} (-1)^{k-1} B_{2k} \frac{2^{2k}}{(2k)!} z^{2k}$$

The result is immediate by comparing the two different representations.

6.4 Exercise

Find the values of

$$\zeta(4)\,,\,\zeta(6)\,,\,B_5\,,\,B_6$$

6.5 Integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

proof

Start by the integral representation

$$\int_0^{\infty} \frac{e^{-t} t^{s-1}}{1 - e^{-t}} \, dt$$

Using the power expansion

$$\frac{1}{1 - e^{-t}} = \sum_{n=0}^{\infty} e^{-nt}$$

Hence we have

$$\int_0^{\infty} e^{-t} t^{s-1} \left(\sum_{n=0}^{\infty} e^{-nt} \right) dt$$

By swapping the series and integral

$$\sum_{n=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-(n+1)t} \, dt = \Gamma(s) \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} = \Gamma(s)\zeta(s)$$

6.6 Hurwitz zeta and polygamma functions

Hurwitz zeta is a generalization of the zeta function by adding a parameter .

6.6.1 Definition

$$\zeta(a,z) \;=\; \sum_{n=0}^{\infty} \frac{1}{(n+z)^a} \;;\zeta(a,1) = \zeta(a)$$

Let us define the polygamma function as the function produced by differentiating the Digamma function and it is often denoted by

$$\psi_n(z) \;\; \forall \; n \geq 0$$

We define the digamma function by setting $n = 0$ so it's denoted by $\psi_0(z)$.

Other values can be found by the following recurrence relation

$$\psi_n'(z) = \psi_{n+1}(z)$$

So we have

$$\psi_1(z) = \psi_0'(z)$$

6.6.2 Relation between zeta and polygamma

$\forall \; n \geq 1$

$$\psi_n(z) \;=\; (-1)^{n+1} n! \, \zeta(n+1, z)$$

$$\psi_{2n-1}(1) = (-1)^{n-1} B_{2n} \frac{2^{2n-2}}{n} \pi^{2n}$$

proof

We have already proved the following relation

$$\psi_0(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$$

This can be written as the following

$$\psi_0(z) = -\gamma + \sum_{k=0}^{\infty} \frac{1}{k+1} - \frac{1}{k+z}$$

By differentiating with respect to z

$$\psi_1(z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^2}$$

$$\psi_2(z) = -2\sum_{k=0}^{\infty} \frac{1}{(k+z)^3}$$

$$\psi_3(z) = 2 \cdot 3 \sum_{k=0}^{\infty} \frac{1}{(k+z)^4}$$

$$\psi_4(z) = -2 \cdot 3 \cdot 4 \sum_{k=0}^{\infty} \frac{1}{(k+z)^5}$$

Continue like that to obtain

$$\psi_n(z) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}}$$

We realize the RHS is just the Hurwitz zeta function

$$\psi_n(z) = (-1)^{n+1} n!\, \zeta(n+1, z)$$

By setting $z = 1$ we have an equation in terms of the ordinary zeta function

$$\psi_n(1) = (-1)^{n+1} n!\, \zeta(n+1)$$

Now since we already proved in the preceding section that

$$\zeta(2k) \;=\; (-1)^{k-1} B_{2k} \frac{2^{2k-1}}{(2k)!} \pi^{2k}$$

we can easily verify the following

$$\psi_{2n-1}(1) = (2n-1)!\, (-1)^{n-1} B_{2n} \frac{2^{2n-1}}{(2n)!} \pi^{2n} = (-1)^{n-1} B_{2n} \frac{2^{2n-2}}{n} \pi^{2n}$$

This can be used to deduce some values for the polygamma function

$$\psi_1(1) = \frac{\pi^2}{6} \;,\; \psi_3(1) = \frac{\pi^4}{15}$$

Other values can be evaluated in terms of the zeta function

$$\psi_2(1) = -2\zeta(3) \;,\; \psi_4(1) = -24\zeta(5)$$

6.7 Example

Prove that

$$\int_0^{\frac{\pi}{2}} x\sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx = \frac{\pi}{16} - \frac{\pi^3}{192}$$

proof

Start by the transformation $x \to \frac{\pi}{2} - x$

$$\int_0^{\frac{\pi}{2}} x\sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx$$

We need to find

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx$$

Let us start by the following

$$F(a,b) = 2\int_0^{\frac{\pi}{2}} \sin^{2a-1}(x)\cos^{2b-1}(x)\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Now let us differentiate with respect to a

$$\frac{\partial}{\partial a}(F(a,b)) = 4\int_0^{\frac{\pi}{2}} \sin^{2a-1}(x)\cos^{2b-1}(x)\log(\sin x)\,dx = \frac{\Gamma(a)\Gamma(b)\left(\psi_0(a)-\psi_0(a+b)\right)}{\Gamma(a+b)}$$

Differentiate again but this time with respect to b

$$\frac{\partial}{\partial b}\left(F_a(a,b)\right) = 8\int_0^{\frac{\pi}{2}} \sin^{2a-1}(x)\cos^{2b-1}(x)\log(\sin x)\ \log(\cos x)dx$$

$$= \frac{\Gamma(a)\Gamma(b)\left(\psi_0^2(a+b)+\psi_0(a)\psi_0(b)-\psi_0(a)\psi_0(a+b)-\psi_0(b)\psi_0(a+b)+\psi_1(a+b)\right)}{\Gamma(a+b)}$$

Putting $a = b = 1$ we have the following

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\ \log(\cos x)dx = \frac{\psi_0^2(2)+\psi_0^2(1)-\psi_0(1)\psi_0(2)-\psi_0(1)\psi(2)-\psi_1(2)}{8}$$

By simple algebra we arrive to

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\ \log(\cos x)dx = \frac{(\psi_0(2)-\psi_0(1))^2-\psi_1(2)}{8}$$

We already know that $\psi_0(1) = -\gamma$ and $\psi_0(2) = 1-\gamma$

Now to evaluate $\psi_1(2)$, we have to use the zeta function we have already established the following relation

$$\psi_1(z) = \sum_{k=0}^{\infty} \frac{1}{(n+z)^2}$$

Now putting $z = 2$ we have the following

$$\psi_1(2) = \sum_{k=0}^{\infty} \frac{1}{(k+2)^2}$$

Let us write the first few terms in the expansion

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

we see this is similar to $\zeta(2)$ but we are missing the first term

$$\psi_1(2) = \zeta(2) - 1 = \frac{\pi^2}{6} - 1$$

Collecting all these results together we have

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\ \log(\cos x)dx\ = \frac{1}{4} - \frac{\pi^2}{48}$$

Finally we get our result

$$\int_0^{\frac{\pi}{2}} x\sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx = \frac{\pi}{16} - \frac{\pi^3}{192}$$

7 Dirichlet eta function

Dirichlet eta function is the alternating form of the zeta function.

7.1 Definition

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

The alternating form of the zeta function is easier to compute once we have established the main results of the zeta function because the alternating form is related to the zeta function through the relation

7.2 Relation to Zeta function

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s)$$

proof

We will start by the RHS

$$\left(1 - 2^{1-s}\right)\zeta(s) = \zeta(s) - 2^{1-s}\zeta(s)$$

Which can be written as sums of series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{2^{s-1}} \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - 2 \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

Clearly we can see that we are subtracting even terms twice , this is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

This looks easier to understand if we write the terms

$$\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots\right) - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right)$$

Rearranging the terms we establish the alternating form

$$1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \eta(s)$$

7.3 Integral representation

$$\eta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t+1}\,dt$$

proof

Start by the RHS

$$\int_0^\infty \frac{t^{s-1}}{e^t+1}\,dt = \int_0^\infty \frac{e^{-t}t^{s-1}}{1+e^{-t}}\,dt$$

Now using the power expansion we arrive to

$$\int_0^\infty e^{-t}t^{s-1}dt\left(\sum_{n=0}^\infty (-1)^n e^{-nt}\right)$$

$$\sum_{n=0}^\infty (-1)^n \int_0^\infty e^{-(n+1)t}t^{s-1}dt$$

Using Laplace transform we can solve the inner integral

$$\int_0^\infty e^{-(n+1)t}t^{s-1}dt = \frac{\Gamma(s)}{(n+1)^s}$$

Hence we have the following

$$\Gamma(s)\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)^s} = \Gamma(s)\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s} = \Gamma(s)\,\eta(s)$$

An easy result of the above integral

$$\int_0^\infty \frac{t}{e^t+1}\,dt = \Gamma(2)\,\eta(2) = \frac{\pi^2}{12}$$

8 Polylogarithm

8.1 Definition

$$\operatorname{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

The name contains two parts, (poly) because we can choose different n and produce many functions and (logarithm) because we can express $\operatorname{Li}_1(z) = -\log(1-z)$.

8.2 Relation to other functions

We can relate it to the Zeta function

$$\operatorname{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n)$$

In particular we have for $n = 2$

$$\operatorname{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$$

Also we can relate it to the eta function though $z = -1$

$$\operatorname{Li}_n(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^n} = -\eta(n)$$

Also we can relate it to logarithms by putting $n = 1$

$$\operatorname{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^k}{k}$$

The power expansion on the left is famous

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z)$$

8.3 Integral representation

$$\operatorname{Li}_{n+1}(z) = \int_0^z \frac{\operatorname{Li}_n(t)}{t}\, dt$$

proof

Using the series representation we have

$$\int_0^z \frac{1}{t}\left(\sum_{k=1}^{\infty} \frac{t^k}{k^n}\right) dt = \sum_{k=1}^{\infty} \frac{1}{k^n} \int_0^z t^{k-1}\, dt = \sum_{k=1}^{\infty} \frac{z^k}{k^{n+1}} = \operatorname{Li}_{n+1}(z)$$

8.4 Square formula

$$\operatorname{Li}_n(-z) + \operatorname{Li}_n(z) = 2^{1-n}\operatorname{Li}_n(z^2)$$

proof

As usual we write the series representation of the LHS

$$\sum_{k=1}^{\infty}\frac{z^k}{k^n} + \sum_{k=1}^{\infty}\frac{(-z)^k}{k^n}$$

Listing the first few terms

$$z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots + \left(-z + \frac{z^2}{2^n} - \frac{z^3}{3^n} + \cdots\right)$$

The odd terms will cancel

$$2\frac{z^2}{2^n} + 2\frac{z^4}{4^n} + 2\frac{z^6}{6^n} + \cdots$$

Take 2^{1-n} as a common factor

$$2^{1-n}\left(z^2 + \frac{(z^2)^2}{2^n} + \frac{(z^2)^3}{3^n} + \cdots\right) = 2^{1-n}\sum_{k=1}^{\infty}\frac{(z^2)^k}{k^n} = 2^{1-n}\operatorname{Li}_n(z^2)$$

8.5 Exercise

Prove that

$$\operatorname{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$$

8.6 Dilogarithms

Of all polylogarithms $\mathrm{Li}_2(z)$ is the most interesting one, in this section we will see why!

8.6.1 Definition

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\log(1-t)}{t}\, dt$$

The curious reader should try to prove the integral representation using the recursive definition we introduced in the previous section .

8.6.2 First functional equation

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) + \mathrm{Li}_2(-z) = -\frac{1}{2}\log^2(z) - \frac{\pi^2}{6}$$

proof

We will start by the following

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) = -\int_0^{\frac{-1}{z}} \frac{\log(1-t)}{t}\, dt$$

Differentiate with respect to z

$$\frac{d}{dz}\mathrm{Li}_2\left(\frac{-1}{z}\right) = \frac{1}{z^2}\left(-\frac{\log\left(1+\frac{1}{z}\right)}{\frac{-1}{z}}\right) = \frac{\log\left(1+\frac{1}{z}\right)}{z} = \frac{\log(1+z)-\log(z)}{z}$$

Now integrate with respect to z

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) = \int_0^{-z} \frac{\log(1-t)}{t}\, dt - \frac{1}{2}\log^2(z) + C = -\mathrm{Li}_2(-z) - \frac{1}{2}\log^2(z) + C$$

To find the constant C let $z = 1$

$$C = 2\mathrm{Li}_2(-1)$$

Now we must be aware that

$$C = 2\mathrm{Li}_2(-1) = -2\eta(2) = \frac{-\pi^2}{6}$$

Which proves the result by simple rearrangement

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) + \mathrm{Li}_2(-z) = -\frac{1}{2}\log^2(z) - \frac{\pi^2}{6}$$

8.6.3 Second functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z) \quad , 0 < z < 1$$

proof

Start by the following

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\, dt$$

Now integrate by parts to obtain

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(t)}{1-t}\, dt - \log(z)\log(1-z)$$

By the change of variable $t = 1-x$ we get

$$\int_0^z \frac{\log(t)}{1-t}\, dt = -\int_1^{1-z} \frac{\log(1-x)}{x}\, dx$$

For $0 < z < 1$

$$\int_{1-z}^1 \frac{\log(1-x)}{x}\, dx = \int_0^1 \frac{\log(1-x)}{x}\, dx - \int_0^{1-z} \frac{\log(1-x)}{x}\, dx$$

Now it is easy to see that

$$\int_{1-z}^1 \frac{\log(1-x)}{x}\, dx = -\mathrm{Li}_2(1) + \mathrm{Li}_2(1-z)$$

Which implies that

$$\mathrm{Li}_2(z) = \mathrm{Li}_2(1) - \mathrm{Li}_2(1-z) - \log(z)\log(1-z)$$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \mathrm{Li}_2(1) - \log(z)\log(1-z)$$

Now since $\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z)$$

We can easily deduce that for $z = \frac{1}{2}$

$$2\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - \log^2\left(\frac{1}{2}\right)$$

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2\left(\frac{1}{2}\right)$$

8.6.4 Third functional equation

$$\operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\log^2(1-z) \quad z < 1$$

proof

Start by the following

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\int_0^{\frac{z}{z-1}} \frac{\log(1-t)}{t}\, dt$$

Differentiate both sides with respect to z

$$\frac{d}{dz}\operatorname{Li}_2\left(\frac{z}{z-1}\right) = \frac{1}{(z-1)^2}\left(\frac{\log\left(1-\frac{z}{z-1}\right)}{\frac{z}{z-1}}\right)$$

Upon simplification we obtain

$$\frac{d}{dz}\operatorname{Li}_2\left(\frac{z}{z-1}\right) = \frac{-\log(1-z)}{z(z-1)}$$

Using partial fractions decomposition

$$\frac{d}{dz}\operatorname{Li}_2\left(\frac{z}{z-1}\right) = \frac{\log(1-z)}{1-z} + \frac{\log(1-z)}{z}$$

Integrate both sides with respect to z

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\log^2(1-z) - \operatorname{Li}_2(z) + C$$

Put $z = -1$ to find the constant

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = -\frac{1}{2}\log^2(2) - \operatorname{Li}_2(-1) + C$$

Remember that

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2\left(\frac{1}{2}\right)\ ,\ \operatorname{Li}_2(-1) = -\frac{\pi^2}{12}$$

Hence we deduce that $C = 0$

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\log^2(1-z) - \operatorname{Li}_2(z)$$

Which can be written as

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) + \operatorname{Li}_2(z) = -\frac{1}{2}\log^2(1-z)$$

8.6.5 Example

Prove that

$$\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$$

proof

First we add the two functional equations of this section to obtain

$$\mathrm{Li}_2\left(\frac{z}{z-1}\right) + \frac{1}{2}\mathrm{Li}_2(z^2) - \mathrm{Li}_2(-z) = -\frac{1}{2}\log^2(1-z)$$

Now let $z = \frac{1-\sqrt{5}}{2}$

$$z^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} \implies \frac{z}{z-1} = \frac{\sqrt{5}-1}{1+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$$

Hence we have

$$\frac{3}{2}\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) - \mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = -\frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right)$$

We already established the following functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z)$$

Put $z = \frac{3-\sqrt{5}}{2}$

$$\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) + \mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{6} - \log\left(\frac{3-\sqrt{5}}{2}\right)\log\left(\frac{\sqrt{5}-1}{2}\right)$$

Solving the two representations for $\mathrm{Li}_2\left(\frac{\sqrt{5}-1}{2}\right)$ we get our result.

8.6.6 Example

Find the following integral

$$I = \int_0^1 \frac{\log(1-x)\log(x)}{x}\,dx$$

Integrate by parts

$$I = -\log(x)\mathrm{Li}_2(x)|_0^1 + \int_0^1 \frac{\mathrm{Li}_2(t)}{t}\,dt = \mathrm{Li}_3(1) = \sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(3)$$

8.6.7 Example

Evaluate the following integral

$$\int_0^x \frac{\log^2(1-t)}{t}\,dt \quad , 0 < x < 1$$

Integrating by parts we get the following

$$\int_0^x \frac{\log^2(1-t)}{t}\,dt = -\log(1-x)\mathrm{Li}_2(x) - \int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\,dt$$

Now we are left with the following integral

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\,dt = \int_{1-x}^1 \frac{\mathrm{Li}_2(1-t)}{t}\,dt$$

Using the first functional equation

$$\int_{1-x}^1 \frac{\frac{\pi^2}{6} - \mathrm{Li}_2(t) - \log(1-t)\log(t)}{t}\,dt$$

$$-\frac{\pi^2}{6}\log(1-x) - \int_{1-x}^1 \frac{\mathrm{Li}_2(t)}{t}\,dt - \int_{1-x}^1 \frac{\log(1-t)\log(t)}{t}\,dt$$

The first integral

$$\int_{1-x}^1 \frac{\mathrm{Li}_2(t)}{t}\,dt = \mathrm{Li}_3(1) - \mathrm{Li}_3(1-x)$$

The second integral is the same as the first exercise

$$\int_{1-x}^1 \frac{\log(1-t)\log(t)}{t} = \mathrm{Li}_3(1) + \log(1-x)\mathrm{Li}_2(1-x) - \mathrm{Li}_3(1-x)$$

Collecting the results together we obtain

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\,dt = -\frac{\pi^2}{6}\log(1-x) - \mathrm{Li}_2(1-x)\log(1-x) + 2\,\mathrm{Li}_3(1-x) - 2\zeta(3)$$

Finally we have

$$\int_0^x \frac{\log^2(1-t)}{t}\,dt = -\log(1-x)\mathrm{Li}_2(x) + \frac{\pi^2}{6}\log(1-x) + \mathrm{Li}_2(1-x)\log(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta(3)$$

8.6.8 Example

Find the following integral

$$I(a) = \int_0^a \frac{x}{e^x - 1}\, dx$$

Start by the power expansion of $\frac{1}{1-e^{-x}}$

$$I(a) = \int_0^a \frac{x}{e^x - 1}\, dx = \int_0^a xe^{-x} \sum_{n=0}^{\infty} e^{-nx}$$

By swapping the integral and the summation

$$I(a) = \sum_{n=0}^{\infty} \int_0^a xe^{-x(n+1)}\, dx$$

The integral could be solved by parts

$$I(a) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} - \frac{e^{-(n+1)a}}{(n+1)^2} - \frac{ae^{-(n+1)a}}{(n+1)}$$

Distribute the summation to obtain

$$I(a) = \zeta(2) + a\log(1 - e^{-a}) - \mathrm{Li}_2(e^{-a})$$

9 Ordinary Hypergeometric function

Ordinary or sometimes called the Gauss hypergoemtric function is a generalization of the power expansion definition. Before we start with the definition we will explain some notations.

9.1 Definition

Define the raising factorial as follows

$$(z)_n = \begin{cases} 1 & : n = 0 \\ \frac{\Gamma(z+n)}{\Gamma(z)} & : n > 0 \end{cases}$$

Using this definition have

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

9.2 Some expansions using the hypergeomtric function

We can represent famous functions using the hypergeomtric function

1. Logarithm

$$z\,{}_2F_1(1,1;2;-z) = z \sum_{n=0}^{\infty} \frac{(1)_n (1)_n}{(2)_n} \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{n!}{(n+1)!} z^{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} = \log(1+z)$$

2. Power function

$${}_2F_1(a,1;1;z) = \sum_{n=0}^{\infty} \frac{(a)_n \, (1)_n}{(1)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}$$

3. Sine inverse

$$z\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; z^2\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} \frac{z^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{2n+1} \frac{z^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \, n! \, (2n+1)} z^{2n+1}$$

   Which can be written as

$$z\,{}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; \tfrac{3}{2}; z^2\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n \, (n!)^2 \, (2n+1)} z^{2n+1} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n \, (2n+1)} z^{2n+1} = \arcsin(z)$$

Now we consider converting the Taylor expansion into the equivalent hypergeomtric representation Suppose the following

$$
{}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} t_k\, z^k \quad , \quad t_0 = 1
$$

Now consider the ratio

$$
\frac{t_{k+1}}{t_k} = \frac{(k+a)(k+b)}{(k+c)(k+1)} z
$$

Using this definition, we can easily find the terms a, b, c.

Let us consider some examples

1. Exponential function

$$
f(z) = e^z
$$

The power expansion is

$$
f(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}
$$

Hence we have

$$
\frac{t_{k+1}}{t_k} = \frac{z}{k+1}
$$

Comparing to our representation we conclude

$$
e^z = {}_2F_1(-,-;-;z)
$$

2. Cosine function

$$
f(z) = \cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}
$$

By the same approach

$$
\frac{t_{k+1}}{t_k} = -\frac{1}{(2k+2)(2k+1)} z = \frac{1}{(k+1)\left(k+\frac{1}{2}\right)} \frac{-z^2}{4}
$$

Hence we have

$$
\cos(z) = {}_2F_1\left(-,-;\frac{1}{2};\frac{-z^2}{4}\right)
$$

3. Power function

$$f(z) = (1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k$$

By the same approach

$$\frac{t_{k+1}}{t_k} = \frac{(k+a)}{(k+1)} z = \frac{(k+a)(k+1)}{(k+1)(k+1)} z$$

Hence we have

$$(1-z)^{-a} = {}_2F_1(a, 1; 1; z)$$

9.3 Exercise

Find the hypergeoemtric representations of the following functions

$$\arcsin(z), \sin(z)$$

9.4 Integral representation

$$\beta(c-b, b)\, {}_2F_1(a, b; c; z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\, dt$$

proof

Start by the RHS

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}\, (1-tz)^{-a}\, dt$$

Using the expansion of $(1-tz)^{-a}$ we have

$$\int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\, (tz)^k$$

Interchanging the integral with the series

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!}\, z^k \int_0^1 t^{k+b-1}(1-t)^{c-b-1}\, dt$$

Recalling the beta function we have

$$\sum_{n=0}^{\infty} \frac{(a)_k \Gamma(k+b)\Gamma(c-b)}{\Gamma(k+c)}\, \frac{z^k}{k!}$$

Using the identity that

$$\beta(c-b,b) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}$$

and

$$\frac{\Gamma(z+k)}{\Gamma(z)} = (z)_k$$

We deduce that

$$\beta(c-b,b)\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(k+b)\Gamma(c)}{\Gamma(b)\Gamma(k+c)}\frac{z^k}{k!} = \beta(c-b,b)\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

9.5 Transformations

1. Pfaff transformations

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

and

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{-b}{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right)$$

proof

Start by the integral representation

$${}_2F_1\left(a,b;c;z\right) = \frac{1}{\beta(b,c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,dt$$

By the the transformation $t \to 1-t$

$$\int_0^1\frac{(1-t)^{b-1}t^{c-b-1}}{(1-(1-t)z)^a}\,dt = \int_0^1\frac{(1-t)^{b-1}t^{c-b-1}}{(1-z+tz)^a}\,dt$$

Which can be written as

$$\frac{(1-z)^{-a}}{\beta(b,c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}\left(1-t\frac{z}{z-1}\right)^{-a}dt$$

Note this is the integral representation of

$$(1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$$

Also using that

$${}_2F_1\left(a, b; c; z\right) = {}_2F_1\left(b, a; c; z\right)$$

We deduce that

$${}_2F_1\left(a, b; c; z\right) = (1-z)^{-b}{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right)$$

2. Euler transformation

$${}_2F_1\left(a, b; c; z\right) = (1-z)^{c-a-b}{}_2F_1\left(c-a, c-b; c; z\right)$$

proof

In the Pfaff transformations let $z \to \frac{z}{z-1}$

$${}_2F_1\left(a, b; c; \frac{z}{z-1}\right) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; z\right)$$

and

$${}_2F_1\left(a, b; c; \frac{z}{z-1}\right) = (1-z)^{-b}{}_2F_1\left(c-a, b; c; z\right)$$

By equating the two transformations

$$(1-z)^{-a}{}_2F_1\left(a, c-b; c; z\right) = (1-z)^{-b}{}_2F_1\left(c-a, b; c; z\right)$$

Now use the transformation $b \to c-b$

$$(1-z)^{-a}{}_2F_1\left(a, b; c; z\right) = (1-z)^{c-b}{}_2F_1\left(c-a, c-b; c; z\right)$$

Which can be reduced to

$${}_2F_1\left(a, b; c; z\right) = (1-z)^{c-a-b}{}_2F_1\left(c-a, c-b; c; z\right)$$

3. Quadratic transformation

$$ {}_2F_1\left(a,b;2b;z\right) = (1-z)^{-\frac{a}{2}}\, {}_2F_1\left(\frac{a}{2}, b-\frac{a}{2}; b+\frac{1}{2}; \frac{z^2}{4z-4}\right) $$

4. Kummer

$$ {}_2F_1\left(a,b;c;z\right) = {}_2F_1\left(a,b;1+a+b-c;1-z\right) $$

9.6 Special values

1. At $z = 1$

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)} $$

Start by the integral representation at $z = 1$

$$ {}_2F_1(a,b;c;1) = \frac{1}{\beta(c-b,b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\, dt $$

Now we can use the first integral representation of the beta function

$$ {}_2F_1(a,b;c;1) = \frac{1}{\beta(c-b,b)} \cdot \frac{\Gamma(b)\Gamma(c-b-a)}{\Gamma(c-a)} $$

Which could be simplified to

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \frac{\Gamma(b)\Gamma(c-b-a)}{\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)} $$

2. At $z = -1$

$$ {}_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)} $$

10 Error Function

The error function is an interesting function that has many applications in probability, statistics and physics.

10.1 Definition

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

10.2 Complementary error function

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

10.3 Imaginary error function

$$\mathrm{erfi}(x) = -i\mathrm{erf}(ix)$$

10.4 Properties

1. The error function is odd

$$\mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\, dt = -\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = -\mathrm{erf}(x)$$

2. Real part and imaginary parts

$$\Re\,\mathrm{erf}(z) = \frac{\mathrm{erf}(z) + \mathrm{erf}(\bar{z})}{2}$$

$$\Im\,\mathrm{erf}(z) = \frac{\mathrm{erf}(z) - \mathrm{erf}(\bar{z})}{2i}$$

Using complex variables it can be done using $\mathrm{erf}(\bar{z}) = \overline{\mathrm{erf}(z)}$

10.5 Relation to other functions

1. Hypergeomtric function

$$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}}\, {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x^2\right)$$

proof

By expanding the hypergeometirc function

$$\frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) = \frac{2x}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k} \frac{(-x^2)^k}{k!}$$

Which can be simplified to

$$\frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x^2\right) = \frac{x}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-x^2)^k}{(\frac{1}{2} + k)k!}$$

Notice that this is actually the expanded error function

$$\frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \int_0^x \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}\, dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-x)^{2k+1}}{(2k+1)k!}$$

2. Incomplete Gamma function

$$\operatorname{erf}(x) = 1 - \frac{\Gamma\left(\frac{1}{2}, x^2\right)}{\sqrt{\pi}}$$

proof

By the definition of the incomplete gamma function

$$\Gamma\left(\frac{1}{2}, x^2\right) = \int_{x^2}^{\infty} t^{-\frac{1}{2}}\, e^{-t}\, dt$$

Let $t = y^2$

$$\Gamma\left(\frac{1}{2}, x^2\right) = 2\int_x^{\infty} e^{-y^2}\, dy$$

We need to get the interval $(0, x)$

$$\Gamma\left(\frac{1}{2}, x^2\right) = 2\int_0^{\infty} e^{-y^2}\, dy - 2\int_0^x e^{-y^2}\, dy$$

We have already proved that

$$2\int_0^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$$

Hence we have using the definition of the error function

$$\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} - \sqrt{\pi}\text{erf}(x)$$

By rearrangements we get our result.

10.6 Example

Find the integral

$$I = \int_0^x e^{t^2}\, dt$$

The function has no elementary anti-derivative so we represent it using the error function.

Consider the imaginary error function

$$\text{erfi}(x) = -i\frac{2}{\sqrt{\pi}}\int_0^{ix} e^{-t^2}\, dt$$

By differentiating both sides we have

$$\frac{d}{dx}\text{erfi}(x) = \frac{2}{\sqrt{\pi}}e^{x^2}$$

Hence we have

$$e^{x^2} = \frac{d}{dx}\frac{\sqrt{\pi}}{2}\text{erfi}(x)$$

By integrating both sides we have

$$\int_0^x e^{t^2}\, dt = \frac{\sqrt{\pi}}{2}\text{erfi}(x)$$

10.7 Example

Prove that

$$\int_0^\infty \text{erfc}(x)\, dx = \frac{1}{\sqrt{\pi}}$$

proof

Using the complementary error function

$$\int_0^\infty (1 - \text{erf}(x))\, dx$$

Integrating by parts we have

$$I = x(1 - \text{erf}(x)) \,]_0^\infty + \frac{2}{\sqrt{\pi}} \int_0^\infty x e^{-x^2}\, dx$$

Now we compute erf(∞)

$$\text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1$$

So the first term will go to zero. The integral can be solved by substitution

$$I = \frac{2}{\sqrt{\pi}} \int_0^\infty x e^{-x^2}\, dx = \frac{1}{\sqrt{\pi}}$$

10.8 Example

Prove that

$$\int_0^\infty \text{erfc}^2(x)\, dx = \frac{2 - \sqrt{2}}{\sqrt{\pi}}$$

proof

Integrate by parts

$$I = x\text{erfc}^2(x) \,]_0^\infty - 2 \int_0^\infty x\text{erfc}'(x)\text{erfc}(x)\, dx$$

The first integral goes to 0

$$I = -2 \int_0^\infty x\, \text{erfc}'(x)\text{erfc}(x)\, dx$$

The derivative of the complementary error function

$$\text{erfc}'(x) = (1 - \text{erf}(x))' = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

That results in

$$I = \frac{4}{\sqrt{\pi}} \int_0^\infty x e^{-x^2} \text{erfc}(x)\, dx$$

Integrate by parts again

$$I = \frac{2}{\sqrt{\pi}} e^{-x^2} \text{erfc}(x) \,]_0^\infty - \frac{4}{\pi} \int_0^\infty e^{-2t^2}\, dt$$

At infinity the integral goes to 0. At 0 we get

$$\frac{2}{\sqrt{\pi}}e^{-x^2}\text{erfc}(x)]_{x=0} = \frac{2}{\sqrt{\pi}}(1 - \text{erf}(0)) = \frac{2}{\sqrt{\pi}}$$

The integral can be evaluated to

$$\int_0^\infty e^{-2t^2}\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Collecting the results together we have

$$I = \frac{2}{\sqrt{\pi}} - \frac{\sqrt{\pi}}{2\sqrt{2}} \times \frac{4}{\pi} = \frac{2 - \sqrt{2}}{\sqrt{\pi}}$$

10.9 Example

Prove that

$$\int_0^\infty \sin(x^2)\text{erfc}(x)\,dx = \frac{\pi - 2\coth^{-1}\sqrt{2}}{4\sqrt{2\pi}}$$

proof

Using the substitution $x = \sqrt{t}$

$$\frac{1}{2}\int_0^\infty \sin(t)t^{-\frac{1}{2}}\text{erfc}(\sqrt{t})\,dt$$

Consider the function

$$I(a) = \frac{1}{2}\int_0^\infty \sin(t)t^{-\frac{1}{2}}\text{erfc}(a\sqrt{t})\,dt$$

Differentiating with respect to a we have

$$I'(a) = \frac{-1}{\sqrt{\pi}}\int_0^\infty \sin(t)e^{-a^2 t}\,dt = \frac{-1}{\sqrt{\pi}} \cdot \frac{1}{a^4 + 1}$$

Now integrating with respect to a

$$I(a) = \frac{-1}{\sqrt{\pi}}\int_0^a \frac{dx}{x^4 + 1} + C$$

To evaluate the constant we take $a \to \infty$

$$I(\infty) = \frac{-1}{\sqrt{\pi}}\int_0^\infty \frac{dx}{x^4 + 1} + C$$

The function has an anti-derivative and the value is

$$\frac{-1}{\sqrt{\pi}}\int_0^\infty \frac{dx}{x^4+1} = -\frac{\sqrt{\pi}}{2\sqrt{2}}$$

Note that

$$\operatorname{erfc}(\infty) = 0 \implies C = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Finally we get

$$I(a) = \frac{-1}{\sqrt{\pi}}\int_0^a \frac{dx}{x^4+1} + \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Let $a = 1$ in the integral

$$I(1) = \int_0^\infty \sin(x^2)\operatorname{erfc}(x)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}} - \frac{1}{\sqrt{\pi}}\int_0^1 \frac{dx}{x^4+1}$$

Also knowing that

$$\frac{1}{\sqrt{\pi}}\int_0^1 \frac{dx}{x^4+1} = \frac{\pi + 2\coth^{-1}\sqrt{2}}{4\sqrt{2\pi}}$$

Hence we have the result

$$\int_0^\infty \sin(x^2)\operatorname{erfc}(x)\,dx = \frac{\pi - 2\coth^{-1}\sqrt{2}}{4\sqrt{2\pi}}$$

10.10 Exercise

Can you find closed forms for

$$\int_0^\infty \operatorname{erfc}^3(x)\,dx = ?$$

$$\int_0^\infty \operatorname{erfc}^4(x)\,dx = ?$$

What about

$$\int_0^\infty \operatorname{erfc}^n(x)\,dx = ?$$

11 Exponential integral function

11.1 Definition

$$E(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = \int_1^\infty \frac{e^{-xt}}{t}\,dt$$

11.2 Example

Prove that

$$\lim_{x\to 0}\left[\log(x) + E(x)\right] = -\gamma$$

proof

Integration by parts for $E(x)$

$$E(x) = e^{-t}\log(t)\;\Big]_x^\infty + \int_x^\infty \log(t)e^{-t}\,dt$$

The limit at infinity goes to zero

$$E(x) = -e^{-x}\log(x) + \int_x^\infty \log(t)e^{-t}\,dt$$

Hence by taking the limit

$$\lim_{x\to 0}\left[\log(x) + E(x)\right] = \lim_{x\to 0}\left(\log(x) - e^{-x}\log(x)\right) + \int_0^\infty \log(t)e^{-t}\,dt$$

The first limit goes to 0

$$\lim_{x\to 0}\left[\log(x) + E(x)\right] = \int_0^\infty \log(t)e^{-t}\,dt = \psi(1) = -\gamma$$

11.3 Example

Prove that for $p > 0$

$$\int_0^\infty x^{p-1}E(ax)\,dx = \frac{\Gamma(p)}{pa^p}$$

proof

Integrating by parts we have

$$\int_0^\infty x^{p-1}E(ax)\,dx = \frac{1}{p}x^pE(ax)\,\Big]_0^\infty + \frac{1}{ap}\int_0^\infty x^{p-1}e^{-ax}\,dx$$

The first limit goes to 0

$$\frac{1}{ap}\int_0^\infty x^{p-1}e^{-ax}\,dx = \frac{1}{pa^p}\int_0^\infty x^{p-1}e^{-x}\,dx = \frac{\Gamma(p)}{p\,a^p}$$

11.4 Example

Prove the general case

$$\int_0^\infty x^{p-1}e^{ax}E(ax)\,dx = \frac{\pi}{\sin(a\pi)}\cdot\frac{\Gamma(p)}{a^p}$$

proof

Switch to the integral representation

$$\int_0^\infty x^{p-1}e^{ax}\int_{ax}^\infty \frac{e^{-t}}{t}\,dt\,dx$$

Use the substitution $t = ax\,y$

$$\int_0^\infty\int_1^\infty x^{p-1}\frac{e^{-ax(y-1)}}{y}\,dy\,dx$$

By switching the two integrals

$$\int_1^\infty \frac{1}{y}\int_0^\infty x^{p-1}e^{-ax(y-1)}dx\,dy$$

By the Laplace identities

$$\frac{\Gamma(p)}{a^p}\int_1^\infty \frac{1}{y(y-1)^p}\,dy$$

Now let $y = 1/x$

$$\frac{\Gamma(p)}{a^p}\int_0^1 x^{p-1}(1-x)^{-p}\,dx$$

Using the reflection formula for the Gamma function

$$\frac{\Gamma(p)}{a^p}\int_0^1 x^{p-1}(1-x)^{-p}\,dx = \frac{\pi}{\sin(a\pi)}\cdot\frac{\Gamma(p)}{a^p}$$

11.5 Example

Prove that

$$\int_0^\infty e^z E^2(z)\,dz = \frac{\pi^2}{6}$$

proof

Using the integral representation

$$E^2(z) = \int_1^\infty \int_1^\infty \frac{e^{-xz}e^{-yz}}{xy}\,dx\,dy$$

$$\int_0^\infty e^z E^2(z)\,dz = \int_0^\infty \int_1^\infty \int_1^\infty \frac{e^{-z(x+y-1)}}{xy}\,dx\,dy\,dz$$

Swap the integrals

$$\int_1^\infty \frac{1}{y} \int_1^\infty \frac{1}{x} \int_0^\infty e^{-z(x+y-1)}\,dz\,dx\,dy$$

$$\int_1^\infty \frac{1}{y} \int_1^\infty \frac{1}{x(x+y-1)}\,dx\,dy$$

The inner integral is an elementary integral

$$\int_1^\infty \frac{1}{x(x+y-1)}\,dx = -\frac{\log(y)}{1-y}$$

The integral becomes

$$\int_1^\infty \frac{\log(y)}{y(y-1)}\,dy$$

Now use the substitution $y = 1/x$

$$-\int_0^1 \frac{\log(x)}{(1-x)}\,dx = -\int_0^1 \frac{\log(1-x)}{x}\,dx = \mathrm{Li}_2(1) = \frac{\pi^2}{6}$$

11.6 Example

Prove that

$$\int_0^\infty z^{p-1} E^2(z)\,dz = \frac{2\Gamma(p)}{p^2}\,{}_2F_1(p,p;p+1;-1)$$

proof

Consider

$$E^2(z) = \int_z^\infty \frac{e^{-t}}{t}\, dt$$

By differentiation with respect to z

$$2E'(z)E(z) = \frac{2\,e^{-z}E(z)}{z}$$

Knowing that we can return to our integral by integration by parts

$$\frac{2}{p}\int_0^\infty z^{p-1}e^{-z}E(z)\,dz$$

Write the integral representation

$$\frac{2}{p}\int_0^\infty z^{p-1}e^{-z}\int_1^\infty \frac{e^{-zt}}{t}\,dtdz$$

Swap the two integrals

$$\frac{2}{p}\int_1^\infty \frac{1}{t}\int_0^\infty z^{p-1}e^{-z(1+t)}\,dz\,dt$$

The inner integral reduces to

$$\frac{2\Gamma(p)}{p}\int_1^\infty \frac{dt}{t(1+t)^p}$$

Use the substitution $t = 1/x$

$$\frac{2\Gamma(p)}{p}\int_0^1 \frac{x^{p-1}}{(1+x)^p}\,dx$$

Using the integral representation of the Hypergeometric function

$$\beta(c-b,b)\,{}_2F_1(a,b;c;z) = \int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-xz)^a}\,dx$$

Let $c = p+1,\ b = p,\ a = p,\ z = -1$

$$\beta(1,p)\,{}_2F_1(p,p;p+1;-1) = \int_0^1 \frac{x^{p-1}}{(1+x)^p}\,dx$$

Hence the result

$$\int_0^\infty z^{p-1}E^2(z)\,dz = \frac{2\Gamma(p)}{p^2}\,{}_2F_1(p,p;p+1;-1)$$

Where

$$\beta(1,p) = \frac{\Gamma(p)}{\Gamma(p+1)} = \frac{1}{p}$$

11.7 Exercise

Find the integral fo $n \in \mathbb{N}$

$$\int_0^\infty x^n E^2(x)\,dx$$

12 Complete Elliptic Integral

12.1 Complete elliptic of first kind

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

12.2 Complete elliptic of second kind

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\, d\theta = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}} dx$$

12.3 Hypergeometric representation

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, k^2\right)$$

and

$$E(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}, 1, k^2\right)$$

proof

Using the integral representation of the hypergeometric function

$$\beta(c-b,b)\, {}_2F_1(a,b,c,z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\, dt$$

Now use the substitution $t = x^2$ and $z = k^2$

$$\beta(c-b,b)\, {}_2F_1(a,b,c,k^2) = 2\int_0^1 \frac{x^{2b-1}(1-x^2)^{c-b-1}}{(1-k^2x^2)^a}\, dx$$

Put $a = \frac{1}{2}$; $b = \frac{1}{2}$ and $c = 1$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} = \frac{1}{2}\beta(1/2, 1/2)\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, k^2\right)$$

By the beta function we have

$$\frac{1}{2}\beta(1/2, 1/2) = \frac{1}{2}\Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

Hence the result

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, k^2\right)$$

By the same approach we have

$$E(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, -\frac{1}{2}, 1, k^2\right)$$

12.4 Example

Prove that

$$\int_0^1 K(k)\,dk = 2G$$

G is the Catalan's constant.

proof

Start by the integral representation

$$I = \int_0^1 \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx\,dk$$

Switching the two integrals

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}} \int_0^1 \frac{1}{\sqrt{1-k^2x^2}}\,dk\,dx$$

$$I = \int_0^1 \frac{\arcsin x}{x\sqrt{1-x^2}} dx$$

Now let $\arcsin x = t$ hence we have $x = \sin t$

$$I = \int_0^{\frac{\pi}{2}} \frac{t}{\sin t} dt$$

The previous integral is a representation of the constant

$$G = \frac{I}{2} \implies I = 2G$$

12.5 Identities

1. For $k \geq 1$

$$K(\sqrt{k}) = \frac{1}{\sqrt{1-k}} K\left(\sqrt{\frac{k}{k-1}}\right)$$

   and

$$E(\sqrt{k}) = \sqrt{1-k}\, E\left(\sqrt{\frac{k}{k-1}}\right)$$

proof

Starting by the integral representation

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

Use the substitution $x = \sqrt{1-y^2}$

$$\int_0^1 \frac{y\, dy}{\sqrt{1-y^2}\sqrt{1-(1-y^2)}\sqrt{1-k^2(1-y^2)}}$$

By cancelling the terms we have

$$\int_0^1 \frac{dy}{\sqrt{1-y^2}\sqrt{1-k^2+k^2y^2}}$$

Take $\sqrt{1-k^2}$ as a common factor

$$\int_0^1 \frac{dy}{\sqrt{1-k^2}\sqrt{1-y^2}\sqrt{1-\frac{k^2}{k^2-1}y^2}}$$

Comparing this to the integral representation we get

$$K(k) = \frac{1}{\sqrt{1-k^2}} K\left(\sqrt{\frac{k^2}{k^2-1}}\right)$$

We can finish by $k \to \sqrt{k}$

$$K(\sqrt{k}) = \frac{1}{\sqrt{1-k}} K\left(\sqrt{\frac{k}{k-1}}\right)$$

Similarly for the second representation

$$E(\sqrt{k}) = \int_0^1 \frac{\sqrt{1-kx^2}}{\sqrt{1-x^2}}\, dx$$

By using that $x = \sqrt{1-y^2}$

$$E(\sqrt{k}) = \sqrt{1-k} \int_0^1 \frac{\sqrt{1-\frac{k}{k-1}y}}{\sqrt{1-y^2}}\, dy = \sqrt{1-k}\, E\left(\sqrt{\frac{k}{k-1}}\right)$$

2.

$$K(k) = \frac{1}{k+1} K\left(\frac{2\sqrt{k}}{1+k}\right)$$

proof

Start by the Quadratic transformation

$$ {}_2F_1\left(a, b, 2b, \frac{4z}{(1+z)^2}\right) = (1+z)^{2a}\, {}_2F_1\left(a, a-b+\frac{1}{2}, b+\frac{1}{2}, z^2\right).$$

Hence we can deduce by putting $a = b = 1/2$

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)K(k)$$

Or we have

$$K(k) = \frac{1}{k+1} K\left(\frac{2\sqrt{k}}{1+k}\right)$$

3.

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1+k}{1-k} K\left(\frac{2\sqrt{-k}}{1-k}\right)$$

and

$$E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1-k}{1+k} E\left(\frac{2\sqrt{-k}}{1-k}\right)$$

proof

Start by the following

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-\frac{4k}{(1+k)^2}x^2}}\, dx$$

By some simplifications we have

$$E\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{(1+k)^2-4k\,x^2}}\, dx$$

Use $x = \sqrt{1-y^2}$

$$\int_0^1 \frac{1+k}{\sqrt{1-y^2}\sqrt{(1+k)^2-4k\,(1-y^2)}}\, dy = \frac{1+k}{1-k}\int_0^1 \frac{1}{\sqrt{1-y^2}\sqrt{1+\frac{4k}{(1-k)^2}y^2}}\, dy$$

Hence we have

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1+k}{1-k} K\left(\frac{2\sqrt{-k}}{1-k}\right)$$

Similarly we have

$$E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1-k}{1+k} E\left(\frac{2\sqrt{-k}}{1-k}\right)$$

12.6 Special values

1.

$$K(i) = \frac{1}{4\sqrt{2\pi}} \Gamma^2\left(\frac{1}{4}\right)$$

and

$$E(i) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

proof

By definition we have

$$K(i) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1+x^2}} = \int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

Let $x = \sqrt[4]{t}$ we have $dx = \frac{1}{4} t^{\frac{-3}{4}}\, dt$

$$K(i) = \frac{1}{4} \int_0^1 t^{\frac{-3}{4}} (1-t)^{\frac{-1}{2}}\, dt$$

By beta function

$$K(i) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{3}{4}\right)}$$

By reflection formula

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi \csc\left(\frac{\pi}{4}\right) = \pi\sqrt{2}$$

$$K(i) = \frac{\Gamma^2\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\pi\sqrt{2}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}}$$

$$E(i) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

By definition we have

$$E(i) = \int_0^1 \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}dx$$

Separating the two integrals

$$E(i) = \int_0^1 \frac{1+x^2}{\sqrt{1-x^4}}dx = \int_0^1 \frac{1}{\sqrt{1-x^4}}dx + \int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx$$

The first integral is $K(i)$ for the second integral use $x = \sqrt[4]{t}$

$$\frac{1}{4}\int_0^1 t^{\frac{3}{4}-1}(1-t)^{-\frac{1}{2}}\,dt = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{5}{4}\right)} = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

Hence we have

$$E(i) = K(i) + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

2.

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

and

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{2\sqrt{\pi}}$$

proof

Start by the identity

$$K(\sqrt{k}) = \frac{1}{\sqrt{1-k}}K\left(\sqrt{\frac{k}{k-1}}\right)$$

For the value $k = -1$

$$K(i) = \frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right)$$

Using the value for $K(i)$

$$K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}K(i) = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

Similarly we have

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{2\sqrt{\pi}}$$

3.

$$K\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{1+\sqrt{2}}{4\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

and

$$E\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{\sqrt{2}}{1+\sqrt{2}}\left[\frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}\right]$$

proof

Start by the identity

$$K(k) = \frac{1}{k+1}K\left(\frac{2\sqrt{k}}{1+k}\right)$$

Hence we have for $k = \frac{1}{\sqrt{2}}$

$$K\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{1+\sqrt{2}}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{2}}{4\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

For the elliptic integral of second kind using the hypergeomtric representation with $a = \frac{-1}{2}$ and $b = \frac{1}{2}$

$${}_2F_1\left(-1/2, 1/2, 1, \frac{4z}{(1+z)^2}\right) = (1+z)^{-1}{}_2F_1\left(-1/2, -1/2, 1, z^2\right)$$

The later hypergeometric series can be written in terms of elliptic integrals using some general contiguity relations

$${}_2F_1\left(-1/2, -1/2, 1, z^2\right) = \frac{2}{\pi}\left(2E(k) + (k^2-1)K(k)\right)$$

So we have

$$2E(k) + (k^2-1)K(k) = (k+1)E\left(\frac{2\sqrt{k}}{1+k}\right)$$

For $k = \frac{1}{\sqrt{2}}$

$$E\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{\sqrt{2}}{1+\sqrt{2}}\left[\frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}\right]$$

4.

$$K\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{\pi\sqrt{\pi}}{4}\cdot\frac{2-\sqrt{2}}{\Gamma^2\left(\frac{3}{4}\right)}$$

and

$$E\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{(2+\sqrt{2})\left(\pi^2+4\Gamma^4\left(\frac{3}{4}\right)\right)}{4\sqrt{\pi}\Gamma^2\left(\frac{3}{4}\right)}$$

proof

Start by the following identity

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1+k}{1-k}K\left(\frac{2\sqrt{-k}}{1-k}\right)$$

Let $x = 1/\sqrt{2}$

$$K\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{\pi\sqrt{\pi}}{4}\cdot\frac{2-\sqrt{2}}{\Gamma^2\left(\frac{3}{4}\right)}$$

$$E\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{(2+\sqrt{2})\left(\pi^2+4\Gamma^4\left(\frac{3}{4}\right)\right)}{4\sqrt{\pi}\Gamma^2\left(\frac{3}{4}\right)}$$

12.7 Differentiation of elliptic integrals

Note We should remove the variable k and denote elliptic integrals E and K once there is no confusion.

It is assumed that the variable is k when we use these symbols.

Interestingly the derivative of elliptic integrals can be written in terms of elliptic integrals

Derivative of complete elliptic integral of second kind

$$\frac{d}{dk}E = \int_0^1 \frac{\frac{\partial}{\partial k}\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx$$

$$\frac{d}{dk}E = \int_0^1 \frac{-k\,x^2}{\sqrt{1-x^2}\sqrt{1-k^2\,x^2}}\,dx$$

Adding and subtracting 1 results in

$$\frac{1}{k}\int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx - \frac{1}{k}\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2\,x^2}}$$

Upon realizing the relation to elliptic integrals we conclude

$$\frac{d}{dk}E = \frac{E-K}{k}$$

For the complete elliptic integral of first kind we need more work

Start by the following

$$\frac{d}{dk}K = \int_0^1 \frac{1}{\sqrt{1-x^2}}\frac{\partial}{\partial k}\left[\frac{1}{\sqrt{1-k^2x^2}}\right]dx$$

$$\frac{d}{dk}K = \int_0^1 \frac{kx^2}{\sqrt{1-x^2}\sqrt{1-k^2x^2}(1-k^2x^2)}\,dx$$

Adding and subtracting 1 we have

$$\frac{-1}{k}\int_0^1 \frac{1-kx^2-1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}(1-k^2x^2)}\,dx = \frac{1}{k}\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}(1-k^2x^2)} - \frac{K}{k}$$

Let us focus on the first integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}(1-k^2x^2)^{\frac{3}{2}}}\,dx$$

Let $x = \sqrt{t}$ and we have $dx = \frac{1}{2\sqrt{t}}\,dt$

$$\frac{1}{2}\int_0^1 \frac{t^{-\frac{1}{2}}}{\sqrt{1-t}(1-k^2t)^{\frac{3}{2}}}\,dx$$

Using the hypergeometric integral representation

$$\frac{1}{2}\int_0^1 \frac{t^{-\frac{1}{2}}}{\sqrt{1-t}(1-k^2t)^{\frac{3}{2}}} = \frac{\pi}{2}{}_2F_1\left(\frac{3}{2},\frac{1}{2},1,k^2\right)$$

Using the linear transformation

$${}_2F_1(a,b,c,z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b,c,z)$$

We get by putting $k' = \sqrt{1-k^2}$

$$\frac{\pi}{2}{}_2F_1\left(\frac{3}{2},\frac{1}{2},1,k^2\right) = \frac{1}{1-k^2}\frac{\pi}{2}{}_2F_1\left(-\frac{1}{2},\frac{1}{2},1,k^2\right) = \frac{E}{k'^2}$$

So finally we get

$$\frac{d}{dk}K = \frac{1}{k}\left(\frac{E}{k'^2}-K\right)$$

13 Euler sums

13.1 Definition

$$S_{p^r,q} = \sum_{k=1}^{\infty} \frac{(H_k^{(p)})^r}{k^q}$$

Where we define the general harmonic number

$$H_k^{(p)} = \sum_{n=1}^{k} \frac{1}{n^p} \ ; \ H_k^{(1)} \equiv H_k = \sum_{n=1}^{k} \frac{1}{n}$$

Euler sums were greatly studied by Euler, hence the name.

13.2 Generating function

$$\sum_{k=1}^{\infty} H_k^{(p)} x^k = \frac{\text{Li}_p(x)}{1-x}$$

Proof

Start by writing $H_k^{(p)}$ as a sum

$$\sum_{k=1}^{\infty} H_k^{(p)} x^k = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{1}{n^p} x^k$$

By interchanging the two series we have

$$\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{x^k}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \sum_{k=n}^{\infty} x^k$$

The inner sum is a geometric series

$$\frac{1}{1-x} \sum_{n=1}^{\infty} \frac{x^n}{n^p} = \frac{\text{Li}_p(x)}{1-x}$$

We can use this to generate some more functions by integrating.

13.3 Integral representation of Harmonic numbers

$$H_n = \int_0^1 \frac{1-x^n}{1-x} \, dx$$

proof

We can use the geometric series of x^n

$$\int_0^1 \frac{1-x^n}{1-x} \, dx = \sum_{k=0}^{\infty} \int_0^1 x^k - x^{n+k} \, dx$$

$$\sum_{k=0}^{\infty} \frac{1}{k+1} - \frac{1}{n+k+1} = H_n$$

13.4 Example

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$$

proof

Using the integral representation

$$\int_0^1 \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{1-x^n}{n^2}\, dx = \int_0^1 \frac{\zeta(2) - \mathrm{Li}_2(x)}{1-x} dx$$

Now use the functional equation

$$\zeta(2) - \mathrm{Li}_2(x) = \mathrm{Li}_2(1-x) + \log(x)\log(1-x)$$

Hence we have

$$\int_0^1 \frac{\mathrm{Li}_2(1-x) + \log(x)\log(1-x)}{1-x} dx$$

The first integral

$$\int_0^1 \frac{\mathrm{Li}_2(1-x)}{1-x} = \mathrm{Li}_3(1) = \zeta(3)$$

The Second integral using integration by parts

$$\int_0^1 \frac{\log(1-x)\log(x)}{x} dx = \int_0^1 \frac{\mathrm{Li}_2(x)}{x} = \zeta(3)$$

Finally we have

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \zeta(3) + \zeta(3) = 2\zeta(3)$$

13.5 Example

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} x^k = \mathrm{Li}_3(x) - \mathrm{Li}_3(1-x) + \log(1-x)\mathrm{Li}_2(1-x) + \frac{1}{2}\log(x)\log^2(1-x) + \zeta(3)$$

proof

In the general definition assume $p = 1$

$$\sum_{k=1}^{\infty} H_k x^k = -\frac{\log(1-x)}{1-x}$$

Divide by x and integrate to get

$$\sum_{k=1}^{\infty} \frac{H_k}{k} x^k = \mathrm{Li}_2(x) + \frac{1}{2}\log^2(1-x)$$

Now divide by x and integrate again

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} x^k = \mathrm{Li}_3(x) + \frac{1}{2}\int_0^x \frac{\log^2(1-t)}{t}\,dt$$

Now let us look at the integral

$$\int_0^x \frac{\log^2(1-t)}{t}\,dt$$

Integrating by parts

$$\int_0^x \frac{\log^2(1-t)}{t}\,dt = -\log(1-x)\mathrm{Li}_2(x) - \int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\,dt$$

Use a change of variable in the integral

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\,dt = \int_{1-x}^1 \frac{\mathrm{Li}_2(1-t)}{t}\,dt$$

Now we can use the second functional equation of the dilogarithm

$$\int_{1-x}^1 \frac{\frac{\pi^2}{6} - \mathrm{Li}_2(t) - \log(1-t)\log(t)}{t}\,dt$$

Separate the integrals

$$-\frac{\pi^2}{6}\log(1-x) - \int_{1-x}^1 \frac{\mathrm{Li}_2(t)}{t}\,dt - \int_{1-x}^1 \frac{\log(1-t)\log(t)}{t}\,dt$$

The first integral

$$\int_{1-x}^1 \frac{\mathrm{Li}_2(t)}{t}\,dt = \mathrm{Li}_3(1) - \mathrm{Li}_3(1-x)$$

Use integration by parts in the second integral

$$\int_{1-x}^1 \frac{\log(1-t)\log(t)}{t} = \mathrm{Li}_3(1) + \log(1-x)\mathrm{Li}_2(1-x) - \mathrm{Li}_3(1-x)$$

Collecting the results together we obtain

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\,dt = -\frac{\pi^2}{6}\log(1-x) - \mathrm{Li}_2(1-x)\log(1-x) + 2\,\mathrm{Li}_3(1-x) - 2\zeta(3)$$

Hence we solved the integral

$$\int_0^x \frac{\log^2(1-t)}{t}\,dt = -\log(1-x)\mathrm{Li}_2(x) + \frac{\pi^2}{6}\log(1-x) + \mathrm{Li}_2(1-x)\log(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta(3)$$

So we have got our Harmonic sum

$$\sum_{k=1}^{\infty}\frac{H_k}{k^2}x^k = \mathrm{Li}_3(x) + \frac{1}{2}\left(-\log(1-x)\mathrm{Li}_2(x) + \frac{\pi^2}{6}\log(1-x) + \mathrm{Li}_2(1-x)\log(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta(3)\right)$$

$$\sum_{k=1}^{\infty}\frac{H_k}{k^2}x^k = \mathrm{Li}_3(x) - \mathrm{Li}_3(1-x) + \log(1-x)\mathrm{Li}_2(1-x) + \frac{1}{2}\log(x)\log^2(1-x) + \zeta(3)$$

13.6 General formula

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q} = \left(1+\frac{q}{2}\right)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k)$$

This can be proved using complex analysis.

13.7 Example

$$\int_0^1 \frac{\log^2(1-x)\log(x)}{x} = -\frac{\pi^4}{180}$$

proof

Using the generating function

$$\sum_{k=1}^{\infty} H_k x^{k-1} = -\frac{\log(1-x)}{x(1-x)}$$

By integrating both sides

$$\sum_{k=1}^{\infty}\frac{H_k}{k}x^k = \mathrm{Li}_2(x) + \frac{1}{2}\log^2(1-x)$$

Or

$$\log^2(1-x) = 2\sum_{k=1}^{\infty}\frac{H_k}{k}x^k - 2\mathrm{Li}_2(x)$$

plugging this in our integral we have

$$2\int_0^1 \left(\sum_{k=1}^\infty \frac{H_k}{k} x^k - \mathrm{Li}_2(x)\right)\frac{\log(x)}{x}\,dx$$

Which simplifies to

$$2\sum_{k=1}^\infty \frac{H_k}{k}\int_0^1 \log(x)x^{k-1}\,dx - 2\int_0^1 \mathrm{Li}_2(x)\frac{\log(x)}{x}\,dx$$

The second integral

$$-2\int_0^1 \mathrm{Li}_2(x)\frac{\log(x)}{x}\,dx = 2\int_0^1 \frac{\mathrm{Li}_3(x)}{x}\,dx = 2\zeta(4)$$

The first integral

$$2\sum_{k=1}^\infty \frac{H_k}{k}\int_0^1 x^{k-1}\log(x)\,dx$$

Using integration by parts twice and the general formula for $q = 3$

$$-2\sum_{k=1}^\infty \frac{H_k}{k^3} = -5\zeta(4) + \zeta^2(2)$$

Finally we get

$$\int_0^1 \frac{\log^2(1-x)\log(x)}{x} = -5\zeta(4) + \zeta^2(2) + 2\zeta(4) = \zeta^2(2) - 3\zeta(4)$$

13.8 Example

Show that

$$\int_0^\infty e^{-at}\sin(bt)\frac{\log t}{t}\,dt = -\left(\frac{\log(a^2+b^2)}{2} + \gamma\right)\arctan\left(\frac{b}{a}\right)$$

Proof

We can start by the following integral

$$I(s) = \int_0^\infty t^{s-1}\,e^{-at}\,\sin(bt)dt$$

By using the the expansion of the sine function

$$I(s) = \int_0^\infty t^{s-1}\,e^{-at}\sum_{n=0}^\infty \frac{(-1)^n(bt)^{2n+1}}{\Gamma(2n+2)}$$

By swapping the summation and integration

$$I(s) = \sum_{n=0}^\infty \frac{(-1)^n (b)^{2n+1}}{\Gamma(2n+2)} \int_0^\infty t^{s+2n}\, e^{-at} dt = \frac{1}{a^s} \sum_{n=0}^\infty \frac{(-1)^n (b)^{2n+1}\Gamma(s+2n+1)}{\Gamma(2n+2)a^{2n+1}}$$

By differentiating and plugging $s = 0$ we have

$$I'(0) = \sum_{n=0}^\infty \frac{(-1)^n \psi_0(2n+1)}{2n+1} \left(\frac{b}{a}\right)^{2n+1} - \log(a) \sum_{n=0}^\infty \frac{(-1)^n}{2n+1} \left(\frac{b}{a}\right)^{2n+1}$$

Now use that $\psi(n+1) + \gamma = H_n$

$$I'(0) = \sum_{n=0}^\infty \frac{(-1)^n H_{2n} - \gamma}{2n+1} \left(\frac{b}{a}\right)^{2n+1} - \log(a) \arctan\left(\frac{b}{a}\right)$$

$$I'(0) = \sum_{n=0}^\infty \frac{(-1)^n H_{2n}}{2n+1} \left(\frac{b}{a}\right)^{2n+1} - (\gamma + \log(a)) \arctan\left(\frac{b}{a}\right)$$

Now we look at the harmonic sum

$$\sum_{k=0}^\infty (-1)^k H_{2k} x^{2k} = \sum_{k=0}^\infty (-1)^k x^{2k} \int_0^1 \frac{1-t^{2k}}{1-t}\, dt$$

Use the integral representation

$$\int_0^1 \frac{1}{1-t} \sum_{k=0}^\infty (-1)^k x^{2k} \left(1 - t^{2k}\right) dt$$

Swap the series and the integral

$$\int_0^1 \frac{1}{1-t} \sum_{k=0}^\infty (-1)^k \left(x^{2k} - (xt)^{2k}\right) dt$$

Evaluate the geometric series

$$\int_0^1 \frac{1}{1-t} \left(\frac{1}{1+x^2} - \frac{1}{1+t^2x^2}\right) dt = \frac{-x^2}{1+x^2} \int_0^1 \frac{(1-t^2)}{(1-t)(1+t^2x^2)}\, dt$$

which simplifies to

$$\frac{-x^2}{1+x^2} \int_0^1 \frac{1+t}{(1+t^2x^2)}\, dt = \frac{-x^2}{1+x^2} \left(\int_0^1 \frac{1}{1+t^2x^2} + \frac{t}{1+t^2x^2}\, dt\right)$$

Evaluating the integrals

$$\frac{-1}{2(1+x^2)} \left(2x \arctan(x) + \log(1+x^2)\right)$$

Using this we conclude by integrating

$$\sum_{k=0}^\infty \frac{(-1)^k H_{2k}}{2k+1} x^{2k} = -\frac{1}{2} \log(1+x^2) \arctan(x)$$

Hence the following

$$\sum_{k=0}^{\infty} \frac{(-1)^k H_{2k}}{2k+1} \left(\frac{b}{a}\right)^{2k+1} = -\frac{1}{2} \log\left(\frac{a^2+b^2}{a^2}\right) \arctan\left(\frac{b}{a}\right)$$

Substituting that in our integral

$$\int_0^\infty e^{-at} \sin(bt) \frac{\log t}{t} \, dt = -\left(\frac{1}{2} \log\left(\frac{a^2+b^2}{a^2}\right) + \gamma + \log(a)\right) \arctan\left(\frac{b}{a}\right)$$
$$= -\left(\frac{\log(a^2+b^2)}{2} + \gamma\right) \arctan\left(\frac{b}{a}\right)$$

13.9 Example

$$\int_0^1 \frac{\mathrm{Li}_p(x)\,\mathrm{Li}_q(x)}{x} \, dx = \sum_{n=1}^{p-1} (-1)^{n-1} \zeta(p-n+1)\zeta(q+n) - \frac{1}{2} \sum_{n=1}^{p+q-2} (-1)^{p-1} \zeta(n+1)\zeta(p+q-n)$$
$$+ (-1)^{p-1} \left(1 + \frac{p+q}{2}\right) \zeta(p+q+1)$$

proof

We can see that

$$\int_0^1 \frac{\mathrm{Li}_p(x)\,\mathrm{Li}_q(x)}{x} \, dx = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{k^q n^p (n+k)}$$

Let us first look at the following

$$\mathscr{C}(\alpha, k) = \sum_{n=1}^{\infty} \frac{1}{n^\alpha (n+k)} \quad ; \quad \mathscr{C}(1, k) = \frac{H_k}{k}$$

This can be solved using

$$\mathscr{C}(\alpha, k) = \sum_{n=1}^{\infty} \frac{1}{k\, n^{\alpha-1}} \left(\frac{1}{n} - \frac{1}{n+k}\right)$$
$$= \frac{1}{k} \zeta(\alpha) - \frac{1}{k} \mathscr{C}(\alpha-1, k)$$
$$= \frac{1}{k} \zeta(\alpha) - \frac{1}{k^2} \zeta(\alpha-1) + \frac{1}{k^2} \mathscr{C}(\alpha-2, k)$$

.

.

.

$$= \frac{1}{k} \zeta(\alpha) - \frac{1}{k^2} \zeta(\alpha-1) + \cdots + (-1)^\alpha \frac{\zeta(2)}{k^{\alpha-1}} + \frac{1}{k^{\alpha-1}} \mathscr{C}(\alpha-(\alpha-1), k)$$
$$= \sum_{n=1}^{\alpha-1} (-1)^{n-1} \frac{\zeta(\alpha-n+1)}{k^n} + (-1)^{\alpha-1} \frac{H_k}{k^\alpha}$$

Hence we have the general formula

$$\mathscr{C}(\alpha,k)=\sum_{n=1}^{\alpha-1}(-1)^{n-1}\frac{\zeta(\alpha-n+1)}{k^n}+(-1)^{\alpha-1}\frac{H_k}{k^\alpha}$$

Dividing by k^β and summing w.r.t to k

$$\sum_{k=1}^\infty\frac{\mathscr{C}(\alpha,k)}{k^\beta}=\sum_{n=1}^{\alpha-1}(-1)^{n-1}\zeta(\alpha-n+1)\zeta(\beta+n)+(-1)^{\alpha-1}\sum_{k=1}^\infty\frac{H_k}{k^{\alpha+\beta}}$$

Now we use the general formula

$$\sum_{n=1}^\infty\frac{H_n}{n^q}=\left(1+\frac{q}{2}\right)\zeta(q+1)-\frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k)$$

Hence we have

$$\sum_{n=1}^\infty\frac{H_n}{n^{\alpha+\beta}}=\left(1+\frac{\alpha+\beta}{2}\right)\zeta(\alpha+\beta+1)-\frac{1}{2}\sum_{k=1}^{\alpha+\beta-2}\zeta(k+1)\zeta(\alpha+\beta-k)$$

And the generalization is the following formula

$$\begin{aligned}\sum_{k=1}^\infty\frac{\mathscr{C}(\alpha,k)}{k^\beta}=&\sum_{n=1}^{\alpha-1}(-1)^{n-1}\zeta(\alpha-n+1)\zeta(\beta+n)-\frac{1}{2}\sum_{n=1}^{\alpha+\beta-2}(-1)^{\alpha-1}\zeta(n+1)\zeta(\alpha+\beta-n)\\&+(-1)^{\alpha-1}\left(1+\frac{\alpha+\beta}{2}\right)\zeta(\alpha+\beta+1)\end{aligned}$$

We conclude by putting that

$$\sum_{k=1}^\infty\sum_{n=1}^\infty\frac{1}{k^qn^p(n+k)}=\sum_{k=1}^\infty\frac{\mathscr{C}(p,k)}{k^q}$$

13.10 Relation to polygamma

We can relate the generalized harmonic number to the polygamma function

$$H_k^{(p)}=\zeta(p)+(-1)^{p-1}\frac{\psi_{p-1}(k+1)}{(p-1)!}$$

proof

$$H_k^{(p)}=\sum_{n=1}^k\frac{1}{n^p}=\zeta(p)-\sum_{n=k+1}^\infty\frac{1}{n^p}$$

Now change the index in the sum $n=i+k+1$

$$H_k^{(p)}=\sum_{n=1}^k\frac{1}{n^p}=\zeta(p)-\sum_{i=0}^\infty\frac{1}{(i+k+1)^p}$$

We know that

$$(-1)^p \frac{\psi_{p-1}(k+1)}{(p-1)!} = \sum_{i=0}^{\infty} \frac{1}{(i+k+1)^p} \quad p \geq 1$$

Hence we have

$$H_k^{(p)} = \zeta(p) + (-1)^{p-1} \frac{\psi_{p-1}(k+1)}{(p-1)!}$$

We can use that to obtain a nice integral representation.

13.11 Integral representation for r=1

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} = \zeta(p)\zeta(q) + (-1)^p \frac{1}{(p-1)!} \int_0^1 \frac{\operatorname{Li}_q(x) \log(x)^{p-1}}{1-x} \, dx$$

proof

Note that

$$\psi_0(a+1) = \int_0^1 \frac{1-x^a}{1-x} \, dx$$

By differentiating with respect to a , p times we have

$$\psi_p(a+1) = \frac{\partial}{\partial a^p} \int_0^1 \frac{1-x^a}{1-x} \, dx$$

$$\psi_p(a+1) = -\int_0^1 \frac{x^a \log(x)^p}{1-x} \, dx$$

Let $a = k$

$$\psi_{p-1}(k+1) = -\int_0^1 \frac{x^k \log(x)^{p-1}}{1-x} \, dx$$

Use the relation to polygamma

$$H_k^{(p)} = \zeta(p) + (-1)^p \frac{1}{(p-1)!} \int_0^1 \frac{x^k \log(x)^{p-1}}{1-x} \, dx$$

Now divide by k^q and sum with respect to k

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} = \zeta(p)\zeta(q) + (-1)^p \frac{1}{(p-1)!} \int_0^1 \frac{\operatorname{Li}_q(x) \log(x)^{p-1}}{1-x} \, dx$$

13.12 Symmetric formula

$$\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q}+\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}=\zeta(p)\zeta(q)+\zeta(p+q)$$

proof

Take the leftmost series and swap the finite and infinite sums

$$\sum_{i=1}^{\infty}\sum_{k=i}^{\infty}\frac{1}{i^p}\frac{1}{k^q}=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{i^p}\frac{1}{k^q}-\sum_{i=1}^{\infty}\frac{1}{i^p}\sum_{k=1}^{i-1}\frac{1}{k^q}$$

The second sum can be written as

$$\sum_{i=1}^{\infty}\frac{1}{i^p}\sum_{k=1}^{i-1}\frac{1}{k^q}=\sum_{i=1}^{\infty}\frac{1}{i^p}\left(\sum_{k=1}^{i}\frac{1}{k^q}-\frac{1}{i^p}\right)$$

By separating and changing the index we get

$$\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}-\zeta(p+q)$$

Hence we have

$$\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q}=\zeta(p)\zeta(q)-\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}+\zeta(p+q)$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q}+\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}=\zeta(p)\zeta(q)+\zeta(p+q)$$

For the special case $p=q=n$

$$\sum_{k=1}^{\infty}\frac{H_k^{(n)}}{k^n}=\frac{\zeta^2(n)+\zeta(2n)}{2}$$

13.13 Example

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^2}=\frac{11\zeta(5)}{2}-2\zeta(2)\zeta(3)$$

proof

Using the symmetric formula

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^2}=\zeta(2)\zeta(3)+\zeta(5)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}$$

Using the integral formula on the second sum

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}=\zeta(2)\zeta(3)+\int_0^1\frac{\operatorname{Li}_3(x)\log(x)}{1-x}\,dx$$

Using integration by parts on the integral

$$\int_0^1 \frac{\mathrm{Li}_3(x)\log(x)}{1-x}\,dx = -\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,dx$$

Let us think of solving

$$\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,dx$$

Using the duplication formula

$$\mathrm{Li}_2(1-x) = \zeta(2) - \mathrm{Li}_2(x) - \log(x)\log(1-x)$$

$$\int_0^1 \frac{\mathrm{Li}_2(x)(\zeta(2) - \mathrm{Li}_2(x) - \log(x)\log(1-x))}{x}\,dx$$

The first integral

$$\zeta(2)\int_0^1 \frac{\mathrm{Li}_2(x)}{x}\,dx = \zeta(2)\zeta(3)$$

The third integral

$$\int_0^1 \frac{\mathrm{Li}_2(x)\log(x)\log(1-x)}{x}\,dx = \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

Finally we get

$$\int_0^1 \frac{\mathrm{Li}_3(x)\log(x)}{1-x}\,dx = \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx - \zeta(2)\zeta(3)$$

So

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k^3} = \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

Hence we finally get that

$$\sum_{k=1}^{\infty} \frac{H_k^{(3)}}{k^2} = \zeta(2)\zeta(3) + \zeta(5) - \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

Let us solve the integral

$$\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

By series expansion

$$\sum_{n=1}^\infty \sum_{k=1}^\infty \frac{1}{(nk)^2} \int_0^1 x^{n+k-1}\, dx = \sum_{n=1}^\infty \sum_{k=1}^\infty \frac{1}{(nk)^2(n+k)}$$

By some manipulations we get

$$\sum_{k=1}^\infty \frac{1}{k^3} \sum_{n=1}^\infty \frac{k}{n^2(n+k)} = \sum_{k=1}^\infty \frac{1}{k^3} \sum_{n=1}^\infty \frac{1}{n^2} - \sum_{k=1}^\infty \frac{1}{k^3} \sum_{n=1}^\infty \frac{1}{n(n+k)}$$

This can be simplified to conclude that

$$\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\, dx = \zeta(2)\zeta(3) - \sum_{k=1}^\infty \frac{H_k}{k^4}$$

Now using that

$$\sum_{k=1}^\infty \frac{H_k}{k^4} = 3\zeta(5) - \zeta(2)\zeta(3)$$

Hence

$$\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\, dx = 2\zeta(2)\zeta(3) - 3\zeta(5)$$

Finally we get

$$\sum_{k=1}^\infty \frac{H_k^{(3)}}{k^2} = \zeta(2)\zeta(3) + \zeta(5) - \frac{3}{2}\left(2\zeta(2)\zeta(3) - 3\zeta(5)\right) = \frac{11\zeta(5)}{2} - 2\zeta(2)\zeta(3)$$

14 Sine Integral function

14.1 Definition

We define the following

$$\mathrm{Si}(z) = \int_0^z \frac{\sin(x)}{x}\,dx$$

A closely related function is the following

$$\mathrm{si}(z) = -\int_z^\infty \frac{\sin(x)}{x}\,dx$$

These functions are related through the equation

$$\mathrm{Si}(z) = \mathrm{si}(z) + \frac{\pi}{2}$$

A closely related function is the sinc function

$$\mathrm{sinc}(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(x)}{x} & x \neq 0 \end{cases}$$

Using that we conclude

$$\frac{d}{dx}\mathrm{Si}(x) = \mathrm{sinc}(x)$$

For integration we have

$$\int \mathrm{Si}(x)\,dx = \cos(x) + x\,\mathrm{Si(x)} + C$$

14.2 Example

Show that

$$\int_0^\infty \sin(x)\mathrm{si}(x)\,dx = -\frac{\pi}{4}$$

proof

Using integration by parts we get

$$-\int_0^\infty \frac{\sin(x)\cos(x)}{x}dx = -\frac{1}{2}\int_0^\infty \frac{\sin(2x)}{x}\,dx$$

Let $2x = t$

$$-\frac{1}{2}\int_0^\infty \frac{\sin(t)}{t}dx = -\frac{\pi}{4}$$

14.3 Example

Prove

$$\int_0^\infty x^{\alpha-1}\,\mathrm{si}(x)\,dx = -\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)$$

proof

Using the integral representation

$$-\int_0^\infty x^{\alpha-1}\int_x^\infty \frac{\sin(t)}{t}\,dt\,dx$$

Let $xy = t$

$$-\int_0^\infty x^{\alpha-1}\int_1^\infty \frac{\sin(xy)}{y}\,dy\,dx$$

Switching the integrals we get

$$-\int_1^\infty \frac{1}{y}\int_0^\infty x^{\alpha-1}\sin(xy)\,dx\,dy$$

Now let $xy = t$

$$-\int_1^\infty \frac{1}{y^{\alpha+1}}\int_0^\infty t^{\alpha-1}\sin(t)\,dt\,dy$$

The Mellin transform of the sine function is defined as

$$\mathcal{M}_s(\sin(x)) = \int_0^\infty x^{s-1}\sin(x)\,dx = \Gamma(s)\sin\left(\frac{\pi s}{2}\right)$$

Hence we conclude that

$$-\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)\int_1^\infty \frac{1}{y^{\alpha+1}} = -\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)$$

14.4 Example

Show that

$$\int_0^\infty e^{-\alpha x}\,\mathrm{si}(x)\,dx = -\frac{\arctan(\alpha)}{\alpha}$$

proof

Use the integral representation

$$-\int_0^\infty e^{-\alpha x}\int_x^\infty \frac{\sin(t)}{t}\,dt\,dx$$

Let $xy = t$

$$-\int_0^\infty e^{-\alpha x}\int_1^\infty \frac{\sin(xy)}{y}\,dy\,dx$$

Switching the integrals

$$-\int_1^\infty \frac{1}{y}\int_0^\infty e^{-\alpha x}\,\sin(xy)\,dx\,dy$$

The inner integral is the laplace transform of the sine function

$$\mathcal{L}_s(\sin(at)) = \frac{a}{s^2 + a^2}$$

Hence we conclude that

$$-\int_1^\infty \frac{1}{y^2+\alpha^2}\,dy = -\frac{\arctan(\alpha)}{\alpha}$$

14.5 Example

Prove the following

$$\int_0^\infty \mathrm{si}(x)\log(x)\,dx = \gamma + 1$$

proof We know that

$$\int_0^\infty x^{\alpha-1}\,\mathrm{si}(x)\,dx = -\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)$$

Differentiate with respect to α

$$\int_0^\infty x^{\alpha-1}\,\mathrm{si}(x)\,\log(x)dx = \frac{\Gamma(\alpha)}{\alpha^2}\sin\left(\frac{\pi\alpha}{2}\right) - \frac{\Gamma(\alpha)\psi(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right) - \frac{\pi\Gamma(\alpha)}{2\alpha}\cos\left(\frac{\pi\alpha}{2}\right)$$

Let $\alpha \to 1$

$$\int_0^\infty \mathrm{si}(x)\,\log(x)dx = 1 - \psi(1) = 1 - (-\gamma) = 1 + \gamma$$

14.6 Example

Find the integral

$$\int_0^\infty \mathrm{si}(x)\,\sin(px)\,dx$$

solution

Using integration by parts we get

$$\left[-\frac{\mathrm{si}(x)\cos(px)}{p}\right]_0^\infty + \frac{1}{p}\int_0^\infty \frac{\sin(x)}{x}\cos(px)\,dx$$

Taking the limits

$$\lim_{x\to 0} -\frac{\mathrm{si}(x)\cos(px)}{p} = \frac{\mathrm{si}(0)}{p} = \frac{\pi}{2p}$$

$$\lim_{x\to\infty} -\frac{\mathrm{si}(x)\cos(px)}{p} = 0$$

Hence we get

$$-\frac{\pi}{2p} + \frac{1}{p}\int_0^\infty \frac{\sin(x)}{x}\cos(px)\,dx$$

The integral

$$\int_0^\infty \frac{\sin(x)}{x}\cos(px)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x) - \sin((p-1)x)}{x}dx$$

Separate the integrals

$$I = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x)}{x}dx - \frac{1}{2}\int_0^\infty \frac{\sin((p-1)x)}{x}dx$$

If $p - 1 > 0$ we get

$$I = \frac{\pi}{4} - \frac{\pi}{4} = 0$$

If $p - 1 < 0$

$$I = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x)}{x}dx + \frac{1}{2}\int_0^\infty \frac{\sin((1-p)x)}{x}dx = \frac{\pi}{2}$$

If $p = 1$ we have

$$I = \frac{1}{2}\int_0^\infty \frac{\sin(2x)}{x}dx + 0 = \frac{\pi}{4}$$

Finally we get

$$\int_0^\infty \mathrm{si}(x)\sin(px)dx = \begin{cases} -\frac{\pi}{2p} & p > 1 \\ -\frac{\pi}{4p} & p = 1 \\ 0 & p < 1 \end{cases}$$

14.7 Example

Prove that for $0 < a < 2$

$$\int_0^\infty \mathrm{si}^2(x)\,\cos(ax)\,dx = \frac{\pi}{2a}\log(a+1)$$

proof

Using integration by parts we get

$$\left[\frac{\mathrm{si}^2(x)\sin(ax)}{a}\right]_0^\infty - \frac{2}{a}\int_0^\infty \frac{\mathrm{si}(x)\sin(x)}{x}\,\sin(ax)\,dx$$

Taking the limits

$$\lim_{x\to 0}\frac{\mathrm{si}^2(x)\sin(ax)}{a} = 0$$

$$\lim_{x\to \infty}\frac{\mathrm{si}^2(x)\sin(ax)}{a} = 0$$

Let the integral

$$I(a) = \int_0^\infty \frac{\mathrm{si}(x)\sin(x)}{x}\,\sin(ax)\,dx$$

Differentiate with respect to a

$$I'(a) = \int_0^\infty \mathrm{si}(x)\sin(x)\,\cos(ax)\,dx$$

Now use the product to sum trigonometric rules

$$I'(a) = \frac{1}{2}\int_0^\infty \mathrm{si}(x)(\sin((a+1)x) - \sin((a-1)x))\,dx$$

From the previous exercise we have

$$\int_0^\infty \mathrm{si}(x)\sin((a+1)x)dx = \frac{-\pi}{4(a+1)} \;\; ; a > 0$$

$$\int_0^\infty \mathrm{si}(x)\sin((a+1)x)dx = 0 \;\; ; a < 2$$

Hence we conclude that for $0 < a < 2$

$$I'(a) = -\frac{\pi}{4(a+1)}$$

Integrate with respect to a

$$I(a) = -\frac{\pi}{4}\log(a+1) + C$$

Let $a \to 0$

$$I(0) = 0 + C \;\; \to \;\; C = 0$$

Hence we have

$$\int_0^\infty \frac{\mathrm{si}(x)\sin(x)}{x}\,\sin(ax)\,dx = -\frac{\pi}{4}\log(a+1)$$

Which implies that

$$\int_0^\infty \mathrm{si}^2(x)\,\cos(ax)\,dx = \frac{-2}{a}\left(-\frac{\pi}{4}\log(a+1)\right) = \frac{\pi}{2a}\log(a+1)$$

14.8 Example

Find the integral, for $a \neq 1$

$$\int_0^\infty \mathrm{si}(x)\cos(ax)\,dx$$

solution

Use integration by parts to obtain

$$\frac{1}{a}\int_0^\infty \frac{\sin(x)\sin(ax)}{x}\,dx$$

Let the integral

$$I(t) = \int_0^\infty e^{-tx}\frac{\sin(x)\sin(ax)}{x}\,dx$$

Differentiate with respect to t

$$I'(t) = -\int_0^\infty e^{-tx}\sin(x)\sin(ax)\,dx$$

Use product to sum rules

$$I'(t) = \frac{1}{2}\int_0^\infty e^{-tx}(\cos((a+1)x) - \cos((a-1)x))\,dx$$

Now we can use the Laplace transform

$$I'(t) = \frac{1}{2}\left(\frac{t}{t^2+(a+1)^2} - \frac{t}{t^2+(a-1)^2}\right)$$

Integrate with respect to t

$$I(t) = -\frac{1}{4}\log\left(\frac{t^2+(a+1)^2}{t^2+(a-1)^2}\right) + C$$

After verifying the constant goes to 0, we have

$$\int_0^\infty e^{-tx}\frac{\sin(x)\sin(ax)}{x}\,dx = -\frac{1}{4}\log\left(\frac{t^2+(a+1)^2}{t^2+(a-1)^2}\right)$$

Let $t \to 0$

$$\int_0^\infty \frac{\sin(x)\sin(ax)}{x}\,dx = -\frac{1}{4}\log\left(\frac{a+1}{a-1}\right)^2$$

We conclude that

$$\int_0^\infty \operatorname{si}(x)\cos(ax)\,dx = -\frac{1}{4a}\log\left(\frac{a+1}{a-1}\right)^2$$

15 Cosine Integral function

15.1 Definition

Define

$$\operatorname{ci}(x) = -\int_x^\infty \frac{\cos(t)}{t}\,dt$$

A related function is the following

$$\operatorname{Cin}(x) = \int_0^x \frac{1-\cos(t)}{t}\,dt$$

The derivative is

$$\frac{d}{dx}\operatorname{ci}(x) = \frac{\cos(x)}{x}$$

The integral

$$\int \operatorname{ci}(x)\,dx = x\operatorname{ci}(x) - \sin(x) + C$$

15.2 Relation to Euler constant

Prove that

$$\lim_{z\to\infty} \operatorname{Cin}(z) - \log z = \gamma$$

proof

Write the integral representation

$$\lim_{z\to\infty} \int_0^z \frac{1-\cos(t)}{t} dt - \log z$$

Can be written

$$\lim_{z\to\infty} \int_0^z \frac{1-\cos(t)}{t} dt - \int_0^z \frac{1}{1+t} dt = \int_0^\infty \frac{1}{t(1+t)} - \frac{\cos(t)}{t} dt$$

This is equivalent to

$$\lim_{s\to 0} \int_0^\infty \frac{t^{s-1}}{(1+t)} - t^{s-1}\cos(t) dt$$

The first integral

$$\int_0^\infty \frac{t^{s-1}}{(1+t)} = \Gamma(s)\Gamma(1-s)$$

The second integral

$$\int_0^\infty t^{s-1}\cos(t)dt = \Gamma(s)\cos(\pi s/2)$$

Hence, it reduces to evaluating the limit

$$\lim_{s\to 0}\Gamma(s)\Gamma(1-s) - \Gamma(s)\cos(\pi s/2)$$

Using $\Gamma(s+1) = s\Gamma(s)$

$$\lim_{s\to 0}\frac{\Gamma(1-s) - \cos(\pi s/2)}{s}$$

Use L'Hospital rule

$$\lim_{s\to 0} -\Gamma(1-s)\psi(1-s) + (\pi/2)\sin(\pi s/2) = -\psi(1) = \gamma$$

15.3 Example

Prove the following

$$\mathrm{Cin}(x) = -\mathrm{ci}(x) + \log(x) + \gamma$$

Start by

$$\mathrm{Cin}(x) = \int_0^x \frac{1-\cos(t)}{t}\,dt$$

Rewrite as

$$\mathrm{Cin}(x) = \int_0^\infty \frac{1-\cos(t)}{t}\,dt - \int_x^\infty \frac{1-\cos(t)}{t}\,dt$$

Which simplifies to

$$\mathrm{Cin}(x) = \lim_{z\to\infty}\left[\int_0^z \frac{1-\cos(t)}{t}\,dt - \log(z)\right] - \mathrm{ci}(x) + \log(x)$$

The limit goes to the Euler Maschorinit constant

$$\mathrm{Cin}(x) = \gamma - \mathrm{ci}(x) + \log(x)$$

15.4 Example

Find the integral

$$\int_0^\infty \mathrm{ci}(x)\ \cos(px)\,dx$$

solution

Using integration by parts we get

$$\left[\frac{\mathrm{ci}(x)\sin(px)}{p}\right]_0^\infty - \frac{1}{p}\int_0^\infty \frac{\cos(x)}{x}\ \sin(px)\,dx$$

Taking the limits

$$\lim_{x\to 0}\frac{\mathrm{ci}(x)\sin(px)}{p} = 0$$

$$\lim_{x\to \infty}\frac{\mathrm{ci}(x)\sin(px)}{p} = 0$$

Hence we get

$$-\frac{1}{p}\int_0^\infty \frac{\cos(x)}{x}\ \sin(px)\,dx$$

The integral

$$\int_0^\infty \frac{\cos(x)}{x}\ \sin(px)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin((p-1)x) + \sin((p+1)x)}{x}dx$$

Separate the integrals

$$I = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x)}{x}dx + \frac{1}{2}\int_0^\infty \frac{\sin((p-1)x)}{x}dx$$

If $p-1>0$ we get

$$I = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

If $p-1<0$

$$I = \frac{\pi}{4} - \frac{\pi}{4} = 0$$

If $p=1$ we have

$$I = \frac{1}{2}\int_0^\infty \frac{\sin(2x)}{x} dx + 0 = \frac{\pi}{4}$$

Finally we get

$$\int_0^\infty \mathrm{ci}(x)\, \cos(px)\, dx = \begin{cases} -\frac{\pi}{2p} & p > 1 \\ -\frac{\pi}{4p} & p = 1 \\ 0 & p < 1 \end{cases}$$

15.5 Example

Find for $p > 1$

$$\int_0^\infty \mathrm{ci}(px)\mathrm{ci}(x)\, dx$$

solution

Let

$$I(p) = \int_0^\infty \mathrm{ci}(px)\mathrm{ci}(x)\, dx$$

Differentiate with respect to p

$$I'(p) = \frac{1}{p}\int_0^\infty \cos(px)\mathrm{ci}(x)\, dx$$

If $p > 1$ from the previous example we conclude that

$$I'(p) = \frac{1}{p}\left(\frac{-\pi}{2p}\right) = -\frac{\pi}{2p^2}$$

Integrate with respect to p

$$I(p) = \frac{\pi}{2p} + C$$

Take the limit $p \to \infty$, so $C = 0$.

15.6 Example

Prove that

$$\int_0^\infty x^{\alpha-1}\mathrm{ci}(x)\, dx = -\frac{\Gamma(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)$$

proof

Use the integral representation

$$-\int_0^\infty x^{\alpha-1}\int_x^\infty \frac{\cos(t)}{t}\,dt\,dx$$

Let $t = yx$

$$-\int_0^\infty x^{\alpha-1}\int_1^\infty \frac{\cos(yx)}{y}\,dy\,dx$$

Switch the integrals

$$-\int_1^\infty \frac{1}{y}\int_0^\infty x^{\alpha-1}\cos(yx)\,dx\,dy$$

Using the Mellin transform we get

$$-\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)\int_1^\infty \frac{1}{y^{1+\alpha}}\,dy = -\frac{\Gamma(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)$$

15.7 Example

Prove that

$$\int_0^\infty \mathrm{ci}(x)\log(x)\,dx = \frac{\pi}{2}$$

proof

From the previous example we know

$$\int_0^\infty x^{\alpha-1}\mathrm{ci}(x)\,dx = -\frac{\Gamma(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)$$

Differentiate with respect to α

$$\int_0^\infty x^{\alpha-1}\mathrm{ci}(x)\log(x)\,dx = \frac{\Gamma(\alpha)}{\alpha^2}\cos\left(\frac{\alpha\pi}{2}\right) - \frac{\Gamma(\alpha)\psi(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) + \frac{\pi}{2}\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)$$

Take the limit $\alpha \to 1$

$$\int_0^\infty \mathrm{ci}(x)\log(x)\,dx = 0 - 0 + \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

15.8 Example

Show that

$$\int_0^\infty \operatorname{ci}(x)e^{-\alpha x}\,dx = -\frac{1}{\alpha}\log\sqrt{1+\alpha^2}$$

proof

Use the integral representation

$$-\int_0^\infty e^{-\alpha x}\int_1^\infty \frac{\cos(yx)}{y}\,dy\,dx$$

Switch the integrals

$$-\int_1^\infty \frac{1}{y}\int_0^\infty e^{-\alpha x}\cos(yx)\,dx\,dy$$

Use the Laplace transformation

$$-\int_1^\infty \frac{\alpha}{y(\alpha^2+y^2)}\,dy = -\frac{1}{2\alpha}\log(1+\alpha^2) = -\frac{1}{\alpha}\log\sqrt{1+\alpha^2}$$

16 Integrals involving Cosine and Sine Integrals

16.1 Example

Find the integral

$$\int_0^\infty \mathrm{si}(qx)\mathrm{ci}(x)\,dx$$

solution

Using the integral representation

$$-\int_0^\infty \mathrm{si}(qx)\int_1^\infty \frac{\cos(yx)}{y}\,dy\,dx$$

Switch the integrals

$$-\int_1^\infty \frac{1}{y}\int_0^\infty \mathrm{si}(qx)\cos(yx)\,dx\,dy$$

We also showed that

$$\int_0^\infty \mathrm{si}(x)\cos(ax)\,dx = -\frac{1}{2a}\log\left(\frac{a+1}{a-1}\right)$$

Let $a = y/q$

$$\int_0^\infty \mathrm{si}(x)\cos(yx/q)\,dx = -\frac{q}{2y}\log\left(\frac{y+q}{y-q}\right)$$

Let $x = tq$

$$\int_0^\infty \mathrm{si}(qt)\cos(yt)\,dx = -\frac{1}{2y}\log\left(\frac{y+q}{y-q}\right)$$

Substitue the value of the itnegral

$$\frac{1}{2}\int_1^\infty \frac{1}{y^2}\log\left(\frac{y+q}{y-q}\right)\,dy$$

We can prove that the anti-derivative

$$\left[\frac{\log(y)}{q} - \frac{1}{2q}\log(y^2-q^2) - \frac{1}{2y}\log\left(\frac{y+q}{y-q}\right)\right]_1^\infty$$

Which simplifies

$$\left[-\frac{1}{2q}\log\left(\frac{y^2-q^2}{y^2}\right) - \frac{1}{2y}\log\left(\frac{y+q}{y-q}\right)\right]_1^\infty$$

The limit $y \to \infty$

$$\lim_{y\to\infty} \frac{1}{2q} \log\left(\frac{y^2 - q^2}{y^2}\right) + \frac{1}{2y} \log\left(\frac{y+q}{y-q}\right) = 0$$

The limit $y \to 1$

$$\frac{1}{2q} \log\left(1 - q^2\right) + \frac{1}{2} \log\left(\frac{1+q}{1-q}\right)$$

Can be written as

$$\frac{1}{4q} \log\left(1 - q^2\right)^2 + \frac{1}{4} \log\left(\frac{1+q}{1-q}\right)^2$$

16.2 Example

Prove that

$$\int_0^\infty \frac{\mathrm{ci}(\alpha x)}{x + \beta}\, dx = -\frac{1}{2}\left\{\mathrm{si}(\alpha\beta)^2 + \mathrm{ci}(\alpha\beta)^2\right\}$$

proof

Let the following

$$I(\alpha) = \int_0^\infty \frac{\mathrm{ci}(\alpha x)}{x + \beta}\, dx$$

Differentiate with respect to α

$$I'(\alpha) = \frac{1}{\alpha} \int_0^\infty \frac{\cos(\alpha x)}{x + \beta}\, dx$$

Let $x + \beta = t$

$$I'(\alpha) = \frac{1}{\alpha} \int_\beta^\infty \frac{\cos(\alpha(t - \beta))}{t}\, dt$$

Use trignometric rules

$$I'(\alpha) = \frac{1}{\alpha} \int_\beta^\infty \frac{\cos(\alpha t)\cos(\alpha\beta) + \sin(\alpha t)\sin(\alpha\beta)}{t}\, dt$$

Separate the integrals

$$I'(\alpha) = \frac{\cos(\alpha\beta)}{\alpha} \int_\beta^\infty \frac{\cos(\alpha t)}{t}\, dt + \frac{\sin(\alpha\beta)}{\alpha} \int_\beta^\infty \frac{\sin(\alpha t)}{t}\, dt$$

This simplifies to

$$I'(\alpha) = -\frac{\cos(\alpha\beta)}{\alpha}\text{ci}(\alpha\beta) - \frac{\sin(\alpha\beta)}{\alpha}\text{si}(\alpha\beta)$$

Integrate with respect to α

$$I(\alpha) = -\frac{1}{2}\left\{\text{si}(\alpha\beta)^2 + \text{ci}(\alpha\beta)^2\right\} + C$$

If $\alpha \to \infty$ we have $C = 0$.

17 Logarithm Integral function

17.1 Definition

Define =

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log(t)}$$

The derivative is

$$\frac{d}{dz}\mathrm{li}(z) = \frac{1}{\log(z)}$$

The integral

$$\int \mathrm{li}(z)\,dz = z\mathrm{li}(z) - \mathrm{Ei}(2\log z)$$

By using integration by parts

$$\int \mathrm{li}(z)\,dz = z\mathrm{li}(z) - \int_0^z \frac{x}{\log(x)}dx$$

In the integral let $-2\log(x) = t$

$$\int \mathrm{li}(z)\,dz = z\mathrm{li}(z) + \int_{-2\log(z)}^{\infty} \frac{e^{-t}}{t}dt = z\mathrm{li}(z) - \mathrm{Ei}(2\log z)$$

17.2 Example

Prove that

$$\int_0^1 \mathrm{li}(x)\,dx = -\log(2)$$

proof

Let the following

$$I(a) = \int_0^1 \int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx$$

Differentiate with respect to a

$$I'(a) = -\int_0^1 \int_0^x t^{-a}\,dt\,dx$$

$$I'(a) = \frac{1}{a-1}\int_0^1 x^{1-a}\,dx$$

Which reduces to

$$I'(a) = \frac{1}{(a-1)(2-a)} = \frac{1}{2-a} - \frac{1}{1-a}$$

Integrate with respect to a

$$I(a) = \log\left(\frac{1-a}{2-a}\right) + C$$

Take the limit $a \to \infty$ we get $C = 0$

$$\int_0^1 \int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx = \log\left(\frac{1-a}{2-a}\right)$$

Let $a \to 0$

$$\int_0^1 \mathrm{li}(x)\,dx = \log\left(\frac{1}{2}\right) = -\log(2)$$

17.3 Find the integral

$$\int_0^1 x^{p-1}\mathrm{li}(x)\,dx$$

solution

Let the following

$$I(a) = \int_0^1 x^{p-1}\int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx$$

Differentiate with respect to a

$$I'(a) = -\int_0^1 x^{p-1}\int_0^x t^{-a}\,dt\,dx$$

$$I'(a) = \frac{1}{a-1}\int_0^1 x^{p-a}\,dx$$

Which reduces to

$$I'(a) = \frac{1}{(a-1)(p-a+1)} = \frac{1}{p}\left\{\frac{1}{p-a+1} - \frac{1}{1-a}\right\}$$

Integrate with respect to a

$$I(a) = \frac{1}{p}\log\left(\frac{1-a}{p-a+1}\right) + C$$

Take the limit $a \to \infty$ we get $C = 0$

$$\int_0^1 x^{p-1} \int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx = \frac{1}{p}\log\left(\frac{1-a}{p-a+1}\right)$$

Let $a \to 0$

$$\int_0^1 x^{p-1} \int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx = \frac{1}{p}\log\left(\frac{1}{p+1}\right) = -\frac{1}{p}\log(p+1)$$

17.4 Find the integral

$$\int_0^1 \mathrm{li}\left(\frac{1}{x}\right)\sin(a\log(x))\,dx$$

proof

Let the following

$$I(b) = \int_0^1 \sin(a\log(x)) \int_0^{\frac{1}{x}} \frac{e^{-b\log(t)}dt}{\log(t)}\,dx$$

Differentiate with respect to b

$$I'(b) = -\int_0^1 \sin(a\log(x)) \int_0^{\frac{1}{x}} t^{-b}\,dt\,dx$$

$$I'(b) = \frac{1}{b-1}\int_0^1 x^{b-1}\sin(a\log(x))\,dx$$

Let $\log(x) = -t$

$$I'(b) = \frac{1}{1-b}\int_0^\infty e^{-tb}\sin(at)\,dt$$

Using the Laplace transform

$$I'(b) = \frac{a}{(b-1)(a^2+b^2)}$$

Integrate with respect to b

$$I(b) = \frac{a\log(a^2+b^2) - a\log(b-1)^2 + 2\arctan(b/a)}{2a^2+2} + C$$

Let $b \to \infty$

$$0 = \frac{\pi}{2(a^2+1)} + C$$

Hence we have

$$\int_0^1 \sin(a\log(x)) \int_0^{\frac{1}{x}} \frac{e^{-b\log(t)}dt}{\log(t)}\, dx = \frac{a\log(a^2+b^2) - a\log(b-1)^2 + 2\arctan(b/a)}{2a^2+2} - \frac{\pi}{2(a^2+1)}$$

Let $b \to 0$

$$\int_0^1 \sin(a\log(x))\mathrm{li}(x)\, dx = \frac{a\log(a^2)}{2a^2+2} - \frac{\pi}{2(a^2+1)} = \frac{1}{a^2+1}\left(a\log(a) - \frac{\pi}{2}\right)$$

17.5 Example

Find the integral

$$\int_0^1 \frac{\mathrm{li}(x)}{x} \log^{p-1}\left(\frac{1}{x}\right) dx$$

proof

Let the following

$$I(a) = \int_0^1 \frac{1}{x}\left[\int_0^x \frac{e^{-a\log(t)}}{\log(t)} dt\right] \log^{p-1}\left(\frac{1}{x}\right) dx$$

Differentiate with respect to a

$$I'(a) = -\int_0^1 \frac{1}{x}\left[\int_0^x t^{-a} dt\right] \log^{p-1}\left(\frac{1}{x}\right) dx$$

$$I'(a) = \frac{1}{a-1}\int_0^1 x^{-a} \log^{p-1}\left(\frac{1}{x}\right) dx$$

Let $-\log(x) = t$

$$I'(a) = \frac{1}{a-1}\int_0^\infty e^{-(1-a)t} t^{p-1}\, dt$$

$$I'(a) = -\frac{\Gamma(p)}{(1-a)(1-a)^p} = -\frac{\Gamma(p)}{(1-a)^{p+1}}$$

Integrate with respect to a

$$I(a) = -\frac{\Gamma(p)}{p(1-a)^p}$$

Let $a \to 0$, Hence

$$\int_0^1 \frac{\operatorname{li}(x)}{x} \log^{p-1}\left(\frac{1}{x}\right) dx = -\frac{\Gamma(p)}{p}$$

17.6 Example

Prove that

$$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \log^{p-1}(x)\, dx = -\frac{\pi}{\sin(\pi p)}\Gamma(p)$$

proof

Let the following

$$I(a) = \int_1^\infty \operatorname{li}\left(x^{-a}\right) \log^{p-1}(x)\, dx$$

Differentiate with respect to a

$$\frac{d}{da}\operatorname{li}\left(x^{-a}\right) = \frac{d}{da}\int_0^{x^{-a}} \frac{dt}{\log(t)} = \frac{x^{-a}}{a}$$

Hence we have

$$I'(a) = \frac{1}{a}\int_1^\infty x^{-a} \log^{p-1}(x)\, dx$$

Let $\log(x) = t$

$$I'(a) = \frac{1}{a}\int_0^\infty e^{-(a-1)t} t^{p-1}\, dt$$

Using the Laplace transform

$$I'(a) = \Gamma(p)\frac{1}{a(a-1)^p}$$

Take the integral

$$\int_1^\infty I'(a) da = \Gamma(p)\int_1^\infty \frac{1}{a(a-1)^p} da$$

The left hand-side

$$I(\infty) - I(1) = \Gamma(p)\int_1^\infty \frac{1}{a(a-1)^p} da$$

Now since $I(\infty) = 0$

$$I(1) = -\Gamma(p) \int_1^\infty \frac{1}{a(a-1)^p} da$$

Which implies that

$$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \log^{p-1}(x)\, dx = -\Gamma(p) \int_1^\infty \frac{1}{a(a-1)^p} da$$

Now let $t = a - 1$

$$\int_0^\infty \frac{t^{-p}}{t+1} dt$$

Using the beta integral $x + y = 1$ and $x - 1 = -p$ which implies that $x = 1 - p, y = p$

Hence we have

$$\int_0^\infty \frac{t^{-p}}{t+1} dt = \beta(p, 1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}$$

Finally we get

$$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \log^{p-1}(x)\, dx = -\frac{\pi}{\sin(\pi p)} \Gamma(p)$$

18 Clausen functions

18.1 Definition

Define

$$\mathrm{cl}_m(\theta) = \begin{cases} \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^m} & m \text{ is even} \\ \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^m} & m \text{ is odd} \end{cases}$$

18.2 Duplication formula

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) - (-1)^m \mathrm{cl}_m(\pi - \theta))$$

proof

If m is even then

$$\mathrm{cli}_m(\pi - \theta) = \sum_{k=1}^{\infty} \frac{\sin(k\pi - k\theta)}{k^m} = -\sum_{k=1}^{\infty} (-1)^k \frac{\sin(k\theta)}{k^m}$$

This implies

$$\mathrm{cl}_2(\theta) + \mathrm{cli}_m(\pi - \theta) = \sum_{k=1}^{\infty} (-1)^k \frac{\sin(k\theta)}{k^m} + \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^m} = \frac{1}{2^{m-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\theta)}{k^m}$$

This implies that

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) - \mathrm{cl}_m(\pi - \theta))$$

If m is odd then

$$\mathrm{cli}_m(\pi - \theta) = \sum_{k=1}^{\infty} \frac{\cos(k\pi - k\theta)}{k^m} = \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\theta)}{k^m}$$

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^m} + \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\theta)}{k^m} = \frac{1}{2^{m-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\theta)}{k^m}$$

Which implies that

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) + \mathrm{cl}_m(\pi - \theta))$$

Collecting the results we have

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) - (-1)^m \mathrm{cl}_m(\pi - \theta))$$

18.3 Example

Find the integral, for m is even

$$\int_0^\pi \mathrm{cl}_m(\theta) d\theta$$

solution

Using the series representation

$$\int_0^\pi \sum_{k=1}^\infty \frac{\sin(k\theta)}{k^m} d\theta$$

Swap the integral and the series

$$\sum_{k=1}^\infty \frac{1}{k^m} \int_0^\pi \sin(k\theta) d\theta$$

The integral

$$\int_0^\pi \sin(k\theta) d\theta = -\left[\frac{1}{k}\cos(k\theta)\right]_0^\pi = \frac{-(-1)^k + 1}{k}$$

We get the summation

$$\sum_{k=1}^\infty \frac{-(-1)^k + 1}{k^{m+1}} = \zeta(m+1) + \eta(m+1)$$

Now use that

$$\eta(s) = (1 - 2^{1-s})\zeta(s)$$

$$\sum_{k=1}^\infty \frac{-(-1)^k + 1}{k^{m+1}} = \zeta(m+1) + (1 - 2^{-m})\zeta(m+1) = \zeta(m+1)(2 - 2^{-m})$$

18.4 Example

Find the integral for m is even

$$\int_0^\infty \mathrm{cl}_m(\theta) e^{-n\theta} d\theta$$

Using the series representation

$$\int_0^\infty \sum_{k=1}^\infty \frac{\sin(k\theta)}{k^m} e^{-n\theta} d\theta$$

Swap the integral and the series

$$\sum_{k=1}^{\infty} \frac{1}{k^m} \int_0^{\infty} \sin(k\theta) e^{-n\theta}\, d\theta$$

Using the Laplace transform we have

$$\sum_{k=1}^{\infty} \frac{1}{k^{m-1}(k^2+n^2)}$$

Add and subtract k^2 and divide by n^2

$$\frac{1}{n^2} \sum_{k=1}^{\infty} \frac{k^2+n^2-k^2}{k^{m-1}(k^2+n^2)}$$

Distribute the numerator

$$\frac{1}{n^2}\zeta(m-1) - \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{1}{k^{m-3}(k^2+n^2)}$$

Continue this approach to conclude that

$$\sum_{l=1}^{j} (-1)^{l-1} \frac{1}{n^{2l}} \zeta(m-(2l-1)) + \frac{(-1)^j}{n^{2j}} \sum_{k=1}^{\infty} \frac{1}{k^{m-(2j+1)}(k^2+n^2)}$$

Let $m - 2j - 1 = 1$ which implies that $j = m/2 - 1$

$$\sum_{l=1}^{m/2-1} (-1)^{l-1} \frac{1}{n^{2l}} \zeta(m-(2l-1)) + \frac{(-1)^{m/2-1}}{n^{m-2}} \sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)}$$

Now let us look at the sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \sum_{k=1}^{\infty} \frac{1}{2ink} \left\{ \frac{1}{k-in} - \frac{1}{k+in} \right\}$$

Which can be written as

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{1}{2n^2} \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \frac{in}{k+in} + \frac{-in}{k-in} \right\}$$

According to the digamma function

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{1}{2n^2} \left\{ \gamma + \psi(1+in) + \psi(1-in) + \gamma \right\}$$

which simplifies to

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{\psi(1-in) + \psi(1+in) + 2\gamma}{2n^2}$$

Now we we can verify $\psi(1-in) = \overline{\psi(1+in)}$

Which suggests that

$$\psi(1+in) + \psi(1-in) = 2\Re\left\{\psi(1+in)\right\}$$

Hence we have the sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{2\Re\left\{\psi(1+in)\right\} + 2\gamma}{2n^2} = \frac{\Re\left\{\psi(1+in)\right\} + \gamma)}{n^2}$$

This concludes to

$$\sum_{l=1}^{m/2-1} (-1)^{l-1} \frac{\zeta(m-(2l-1))}{n^{2l}} + (-1)^{m/2-1} \frac{\Re\left\{\psi(1+in)\right\} + \gamma}{n^m}$$

19 Clausen Integral function

19.1 Definiton

We define

$$\mathrm{cl}_2(x) = \sum_{k=1} \frac{\sin(kx)}{k^2}$$

19.2 Integral representation

$$\mathrm{cl}_2(\theta) = -\int_0^\theta \log\left[2\sin\left(\frac{\phi}{2}\right)\right] d\phi$$

proof

Start by the following

$$\mathrm{Li}_2(e^{i\theta}) = \sum_{k=1}^\infty \frac{e^{ik\theta}}{k^2} = \sum_{k=1}^\infty \frac{\cos(k\theta)}{k^2} + i\sum_{k=1}^\infty \frac{\sin(k\theta)}{k^2}$$

By the integral definition of the dilogarithm

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -\int_1^{e^{i\theta}} \frac{\log(1-x)}{x}\, dx$$

Let $x = e^{i\phi}$

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -i\int_0^\theta \log(1-e^{i\phi}) d\phi$$

Let us look at the following

$$1 - e^{i\phi} = 1 - \cos(\phi) - i\sin(\phi) = 2\sin^2(\phi/2) - 2i\sin(\phi/2)\cos(\phi/2)$$

Which simplifies to

$$1 - e^{i\phi} = 2\sin(\phi/2)\left[\sin(\phi/2) - i\cos(\phi/2)\right] = 2\sin(\phi/2)e^{-(i/2)(\pi-\phi)}$$

Hence our integral

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -i\int_0^\theta \log\left[2\sin(\phi/2)e^{-(i/2)(\pi-\phi)}\right] d\phi$$

Use the complex logarithm properties

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -i\int_0^\theta \log\left[2\sin(\phi/2)\right] d\phi + \frac{1}{4}(\pi-\theta)^2 - \frac{1}{4}\pi^2$$

By equating the imaginary parts we have our result.

We can see the special value

$$\mathrm{cl}_2\left(\frac{\pi}{2}\right) = \sum_{k=1}^{\infty} \frac{\sin(k\pi/2)}{k^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = G$$

Where G is the Catalan's constant.

19.3 Duplication formula

Prove the following

$$\mathrm{cl}_2(2\theta) = 2(\mathrm{cl}_2(\theta) - \mathrm{cl}_2(\pi - \theta))$$

proof

We provide a proof using the integral representation

$$\mathrm{cl}_2(2\theta) = -\int_0^{2\theta} \log\left[2\sin\left(\frac{t}{2}\right)\right] dt$$

Let $t = 2\phi$

$$-2\int_0^{\theta} \log\left[2\sin(\phi)\right] d\phi$$

Use the double angle identity

$$-2\int_0^{\theta} \log\left[4\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\right] d\phi$$

Separate the logarithms

$$-2\int_0^{\theta} \log\left[2\sin\left(\frac{\phi}{2}\right)\right] d\phi - 2\int_0^{\theta} \log\left[2\cos\left(\frac{\phi}{2}\right)\right] d\phi$$

We can verify that

$$\mathrm{cl}_2(\pi - \theta) = \int_0^{\theta} \log\left[2\cos\left(\frac{\phi}{2}\right)\right] d\phi$$

Hence

$$\mathrm{cl}_2(2\theta) = 2(\mathrm{cl}_2(\theta) - \mathrm{cl}_2(\pi - \theta))$$

Using that we get the value

$$\mathrm{cl}_2(3\pi) = 2\mathrm{cl}_2\left(\frac{3\pi}{2}\right) - 2\mathrm{cl}_2\left(-\frac{\pi}{2}\right)$$

Since $\mathrm{cl}_2(3\pi) = 0$

$$\mathrm{cl}_2\left(\frac{3\pi}{2}\right) = \mathrm{cl}_2\left(-\frac{\pi}{2}\right) = -\mathrm{cl}_2\left(\frac{\pi}{2}\right) = -G$$

19.4 Example

Prove that

$$\int_0^{2\pi} \mathrm{cl}_2(x)^2 dx = \frac{\pi^5}{90}$$

Using the series representation

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(nk)^2} \int_0^{2\pi} \sin(kx)\sin(nx)dx$$

Consider the integral

$$\int_0^{2\pi} \sin(kx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \cos((k-n)x) - \cos((k+n)x)\,dx$$

We have two cases

If $n = k$ then

$$\frac{1}{2}\int_0^{2\pi} 1 - \cos(2nx)\,dx = \pi$$

If $n \neq k$

$$\frac{1}{2}\int_0^{2\pi} \cos((k-n)x) - \cos((k+n)x)\,dx = \frac{1}{2}\left[\frac{\sin((k-n)x)}{k-n} - \frac{\sin((k+n)x)}{k+n}\right]_0^{2\pi} = 0$$

Hence we have

$$\int_0^{2\pi} \sin(kx)\sin(nx)\,dx = \begin{cases} 0 & n \neq k \\ \pi & n = k \end{cases}$$

We can write the series as

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(nk)^2} = \sum_{n\neq k}^{\infty} \frac{1}{(nk)^2} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Now since the integral $n \neq k$ goes to zero the result reduces to

$$\pi \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi\zeta(4) = \frac{\pi^5}{90}$$

19.5 Example

Prove that

$$\int_0^{\pi/2} x \log(\sin x)\, dx = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8} \log 2$$

proof

$$I = \int_0^{\frac{\pi}{2}} x \log(\sin x)\, dx = \int_0^{\frac{\pi}{2}} x \log(2 \sin x) - \frac{\pi^2}{8} \log(2)$$

The integral reduces to

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x \log(2 \sin x) dx &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(2\theta)\, d\theta \\
&= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n^2}\, d\theta \\
&= -\frac{1}{4} \sum_{n=1} \frac{(-1)^n}{n^3} + \frac{1}{4} \sum_{n=1} \frac{1}{n^3} \\
&= \frac{7}{16}\zeta(3)
\end{aligned}$$

Collecting that together we have

$$I = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8} \log(2)$$

19.6 Example

Prove that

$$\int_0^{\pi/4} x \cot(x)\, dx = -\frac{\pi}{8} \log(2) + G/2$$

proof

Start by integration by parts

$$\int_0^{\pi/4} x \cot(x)\, dx = \frac{\pi}{8} \log(2) - \int_0^{\pi/4} \log(\sin x)\, dx$$

In the integral

$$\int_0^{\pi/4} \log(\sin x)\, dx$$

Let $x \to t/2$

$$\frac{1}{2}\int_0^{\pi/2} \log(\sin t/2)\, dt$$

Which can be written as

$$\frac{1}{2}\int_0^{\pi/2} \log(2\sin t/2)\, dt - \frac{1}{2}\int_0^{\pi/2} \log(2)\, dt$$

Using the Clausen integral function we get

$$-\frac{1}{2}\mathrm{cl}_2(\pi/2) - \frac{\pi}{4}\log(2)$$

Note that $\mathrm{cl}_2(\pi/2) = G$

We deduce that

$$\int_0^{\pi/4} \log(\sin x)\, dx = -G/2 - \frac{\pi}{4}\log(2)$$

Collecting the results we have

$$\int_0^{\pi/4} x\cot(x)\, dx = \frac{\pi}{8}\log(2) + G/2 - \frac{\pi}{4}\log(2) = -\frac{\pi}{8}\log(2) + G/2$$

19.7 Second Integral representation

Prove that

$$\mathrm{cl}_2(\theta) = -\sin(\theta)\int_0^1 \frac{\log(x)}{x^2 - 2\cos(\theta)x + 1} dx$$

proof

Note that

$$x^2 - 2\cos(\theta)x + 1 = x^2 - (e^{i\theta} + e^{-i\theta})x + 1 = (x - e^{i\theta})(x - e^{-i\theta})$$

This implies

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{1}{e^{i\theta} - e^{-i\theta}}\left\{\frac{1}{x - e^{i\theta}} - \frac{1}{x - e^{-i\theta}}\right\}$$

Note that $e^{i\theta} - e^{-i\theta} = 2i\sin(\theta)$

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{1}{2i\sin(\theta)}\left\{\frac{1}{x - e^{i\theta}} - \frac{1}{x - e^{-i\theta}}\right\}$$

Now use the geometric series

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{-1}{2i\sin(\theta)}\left\{\sum_{k=0}^{\infty} x^k e^{-i(k+1)\theta} - \sum_{k=0}^{\infty} x^k e^{i(k+1)\theta}\right\}$$

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{1}{\sin(\theta)}\sum_{k=1}^{\infty} x^{k-1}\sin(k\theta)$$

That implies

$$-\sin(\theta)\int_0^1 \frac{\log(x)}{x^2 - 2\cos(\theta)x + 1}dx = -\sum_{k=1}^{\infty}\sin(k\theta)\int_0^1 x^{k-1}\log(x)\,dx = \mathrm{cl}_2(\theta)$$

19.8 Example

Find the value of

$$\mathrm{cl}_2\left(\frac{2\pi}{3}\right)$$

proof

Use the second integral representation

$$\mathrm{cl}_2(2\pi/3) = -\frac{\sqrt{3}}{2}\int_0^1 \frac{\log(x)}{x^2 + x + 1}dx$$

Use that

$$x^3 - 1 = (x-1)(x^2 + x + 1)$$

Hence

$$\mathrm{cl}_2(2\pi/3) = -\frac{\sqrt{3}}{2}\int_0^1 \frac{x-1}{x^3-1}\log(x)dx$$

Let $x^3 = t$

$$\mathrm{cl}_2(2\pi/3) = -\frac{1}{6\sqrt{3}}\int_0^1 \frac{t^{1/3-1}(t^{1/3}-1)}{t-1}\log(t)dx$$

Note that

$$\psi'(s) = \int_0^1 \frac{x^{s-1}}{1-x} \log(x) \, dx$$

We deduce that

$$\mathrm{cl}_2(2\pi/3) = -\frac{1}{6\sqrt{3}} \left(\psi'(2/3) - \psi'(1/3)\right)$$

20 Barnes G function

20.1 Definition

$$G(z+1) = (2\pi)^{z/2} \exp\left(-\frac{z + z^2(1+\gamma)}{2}\right) \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right)^n \exp\left(\frac{z^2}{2n} - z\right) \right\}$$

20.1.1 Functional equation

Prove that

$$G(z+1) = \Gamma(z) G(z)$$

proof

From the series representation we have

$$\frac{G(z+1)}{G(z)} = \sqrt{2\pi} \exp\left(-z - \gamma z + \frac{\gamma}{2}\right) \prod_{k=1}^{\infty} \left(\frac{k+z}{k+z-1}\right)^k \exp\left(\frac{2z-1-2k}{2k}\right).$$

This can be written as

$$\frac{G(z+1)}{G(z)} = z\sqrt{2\pi} \exp\left(-z + \frac{\gamma}{2}\right) \prod_{k=1}^{\infty} \left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right) \left(1 + \frac{z}{k}\right) \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{\left(\frac{z}{k}\right)}$$

Use the definition of the gamma function

$$\frac{G(z+1)}{G(z)} = z\Gamma(z)\sqrt{2\pi} \exp\left(-z + \frac{\gamma}{2}\right) \prod_{k=1}^{\infty} \left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right) \left(1 + \frac{z}{k}\right)$$

It suffices to prove that

$$z\sqrt{2\pi} \exp\left(-z + \frac{\gamma}{2}\right) \prod_{k=1}^{\infty} \left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right) \left(1 + \frac{z}{k}\right) = 1$$

or

$$\prod_{k=1}^{\infty} \left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right) \left(1 + \frac{z}{k}\right) = \frac{\exp\left(z - \frac{\gamma}{2}\right)}{z\sqrt{2\pi}}$$

Start by

$$\lim_{N \to \infty} \prod_{k=1}^{N} \left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right) \left(1 + \frac{z}{k}\right)$$

Notice

$$\prod_{k=1}^{N}\left(\frac{k+z}{k+z-1}\right)^k\left(1+\frac{z}{k}\right)=\frac{\prod_{k=1}^{N}(k+z)^k\prod_{k=1}^{N}\left(1+\frac{z}{k}\right)}{\prod_{k=1}^{N}(k+z-1)^k}$$

$$=\frac{\prod_{k=1}^{N}(k+z)^k\prod_{k=1}^{N}(k+z)}{zN!\prod_{k=1}^{N-1}(k+z)^{k+1}}$$

$$=\frac{(N+z)^{N+1}\prod_{k=1}^{N-1}(k+z)^k\prod_{k=1}^{N-1}(k+z)}{zN!\prod_{k=1}^{N-1}(k+z)^{k+1}}$$

$$=\frac{(N+z)^{N+1}\prod_{k=1}^{N-1}(k+z)^{k+1}}{zN!\prod_{k=1}^{N-1}(k+z)^{k+1}}$$

$$=\frac{(N+z)^{N+1}}{zN!}$$

The second product

$$\prod_{k=1}^{N}\exp\left(-\frac{1+2k}{2k}\right)=\exp\left(-\sum_{k=1}^{N}\frac{1+2k}{2k}\right)=e^{-\frac{1}{2}H_N-N}$$

Hence we have the following

$$e^{-\frac{1}{2}H_N-N}\frac{(N+z)^{N+1}}{zN!}$$

According to Stirling formula we have

$$e^{-\frac{1}{2}H_N-N}\frac{(N+z)^{N+1}}{zN!}\sim e^{-\frac{1}{2}H_N-N}\frac{(N+z)^{N+1}}{z(N/e)^N}\times\frac{1}{\sqrt{2\pi N}}$$

By some simplifications we have

$$\frac{e^{-\frac{1}{2}(H_N-\log N)}}{z}\left(1+\frac{z}{N}\right)\times\left(1+\frac{z}{N}\right)^N\times\frac{1}{\sqrt{2\pi}}\sim\frac{\exp\left(-\frac{\gamma}{2}+z\right)}{z\sqrt{2\pi}}$$

Where we used that

$$\lim_{n\to\infty}H_n-\log(n)=\gamma$$

and

$$\lim_{n\to\infty}\left(1+\frac{z}{N}\right)^N=e^z$$

20.2 Reflection formula

$$\log\left\{\frac{G(1-z)}{G(1+z)}\right\}=z\log\left(\frac{\sin(\pi z)}{\pi}\right)+\frac{\mathrm{cl}_2(2\pi z)}{2\pi}$$

proof

Start by the series expansion

$$\frac{G(1-z)}{G(1+z)} = \frac{(2\pi)^{-z/2}\exp\left(\frac{z-z^2(1+\gamma)}{2}\right)\prod_{n=1}^{\infty}\left\{\left(1-\frac{z}{n}\right)^n \exp\left(\frac{z^2}{2n}+z\right)\right\}}{(2\pi)^{z/2}\exp\left(-\frac{z+z^2(1+\gamma)}{2}\right)\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^n \exp\left(\frac{z^2}{2n}-z\right)\right\}}$$

This simplifies to

$$\frac{G(1-z)}{G(1+z)} = (2\pi)^{-z}e^z\prod_{n=1}^{\infty}\frac{(n-z)^n}{(n+z)^n}e^{2z}$$

Take the log of both sides

$$\log\left\{\frac{G(1-z)}{G(1+z)}\right\} = -z\log(2\pi) + z + \log\left\{\prod_{n=1}^{\infty}\frac{(n-z)^n}{(n+z)^n}e^{2z}\right\}$$

Let the following

$$f(z) = \log\left\{\prod_{n=1}^{\infty}\frac{(n-z)^n}{(n+z)^n}e^{2z}\right\} = \sum_{n=1}^{\infty} n\log(n-z) - n\log(n+z) + 2z$$

Differentiate with respect to z

$$f'(z) = \sum_{n=1}^{\infty}\frac{-n}{n-z} - \frac{n}{n+z} + 2 = \sum_{n=1}^{\infty}\frac{-n(n+z) - n(n-z) + 2(n^2-z^2)}{n^2-z^2}$$

Hence we have

$$\sum_{n=1}^{\infty}\frac{-2z^2}{n^2-z^2}$$

Now we can use the following

$$z\pi\cot\pi z = 1 + \sum_{n=1}^{\infty}\frac{-2z^2}{n^2-z^2}$$

Hence we conclude that

$$f'(z) = z\pi\cot\pi z - 1$$

Integrate with respect to z

$$f(z) = \int_0^z x\pi\cot(\pi x)\,dx - z$$

Hence we have

$$\log\left\{\frac{G(1-z)}{G(1+z)}\right\} = -z\log(2\pi) + \int_0^z z\pi\cot(\pi x)\,dx$$

Now use integration by parts for the integral

$$\int_0^z x\pi\cot(\pi x)\,dx = z\log(\sin\pi z) - \int_0^z \log(\sin\pi x)dx \quad (1)$$

$$= z\log(2\sin\pi z) - \int_0^z \log(2\sin\pi x)dx \quad (2)$$

$$= z\log(2\sin\pi z) - \frac{1}{2\pi}\int_0^{2\pi z} \log\left(2\sin\frac{x}{2}\right)dx \quad (3)$$

$$= z\log(2\sin\pi z) + \frac{\mathrm{cl}_2(2\pi z)}{2\pi} \quad (4)$$

$$(5)$$

That implies

$$\log\left\{\frac{G(1-z)}{G(1+z)}\right\} = z\log(2\sin\pi z) - z\log(2\pi) + \frac{\mathrm{cl}_2(2\pi z)}{2\pi} = z\log\left(\frac{\sin(\pi z)}{\pi}\right) + \frac{\mathrm{cl}_2(2\pi z)}{2\pi}$$

20.3 Values at positive integers

Prove that

$$G(n) = \prod_{k=1}^{n-1}\Gamma(k)$$

proof

It can be proved by induction. For $G(1) = 1$, suppose

$$G(n) = \prod_{k=1}^{n-1}\Gamma(k)$$

We want to show

$$G(n+1) = \prod_{k=1}^{n}\Gamma(k)$$

By the functional equation

$$G(n+1) = \Gamma(n)G(n) = \Gamma(n)\prod_{k=1}^{n-1}\Gamma(k) = \prod_{k=1}^{n}\Gamma(k)$$

20.4 Relation to Hyperfactorial function

We define the hyperfactorial function as

$$H(n) = \prod_{k=1}^{n} k^k$$

Prove for n is a positive integer

$$G(n+1) = \frac{(n!)^n}{H(n)}$$

proof

We can prove it by induction for $n = 0$ we have, $G(1) = 1$,

suppose that

$$G(n) = \frac{\Gamma(n)^{n-1}}{H(n-1)}$$

we want to show that

$$G(n+1) = \Gamma(n)G(n) = \Gamma(n)\frac{\Gamma(n)^{n-1}}{H(n-1)}$$

Notice that

$$H(n-1) = \prod^{n-1} k^k = \frac{\prod^n k^k}{n^n} = \frac{H(n)}{n^n}$$

We deduce that

$$G(n+1) = \Gamma(n)G(n) = \frac{\Gamma(n)^n \times n^n}{H(n)} = \frac{(n!)^n}{H(n)}$$

20.5 Loggamma integral

Prove that

$$\int_0^z \log\Gamma(x)dx = \frac{z}{2}\log(2\pi) + \frac{z(z-1)}{2} + z\log\Gamma(z) - \log G(z+1)$$

proof

Take the log to the series representation

$$\log G(z+1) = \frac{z}{2}\log(2\pi) - \frac{z+z^2(1+\gamma)}{2} + \sum_{n=1}^{\infty} n\log\left(1+\frac{z}{n}\right) + \frac{z^2}{2n} - z$$

Let the following

$$f(z) = \sum_{n=1}^{\infty} n \log\left(1 + \frac{z}{n}\right) + \frac{z^2}{2n} - z$$

Differentiate with respect to z

$$f'(z) = \sum_{n=1}^{\infty} \frac{n}{z+n} + \frac{z}{n} - 1 = \sum_{n=1}^{\infty} \frac{z^2}{n(n+z)}$$

Now use the following

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$$

which implies that

$$\sum_{n=1}^{\infty} \frac{z^2}{n(n+z)} = z\psi(z) + \gamma z + 1$$

Hence we have

$$f'(z) = z\psi(z) + \gamma z + 1$$

Integrate with respect to z

$$f(z) = \int_0^z x\psi(x)dx + \frac{\gamma z^2}{2} + z$$

which implies that

$$f(z) = z \log\Gamma(z) - \int_0^z \log\Gamma(x)dx + \frac{\gamma z^2}{2} + z$$

Hence we have

$$\log G(z+1) = \frac{z}{2}\log(2\pi) - \frac{z + z^2(1+\gamma)}{2} + z\log\Gamma(z) - \int_0^z \log\Gamma(x)dx + \frac{\gamma z^2}{2} + z$$

By some rearrangements we have

$$\int_0^z \log\Gamma(x)dx = \frac{z}{2}\log(2\pi) + \frac{z(z-1)}{2} + z\log\Gamma(z) - \log G(z+1)$$

20.6 Glaisher-Kinkelin constant

We define the Glaisher-Kinkelin constant as

$$A = \lim_{n\to\infty} \frac{H(n)}{n^{n^2/2+n/2+1/12}e^{-n^2/4}}$$

20.7 Relation to Glaisher-Kinkelin constant

Prove that

$$\lim_{n\to\infty} \frac{G(n+1)}{(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}} = \frac{e^{1/12}}{A}$$

proof

Use the relation to the hyperfactorial function

$$\lim_{n\to\infty} \frac{(n!)^n}{H(n)(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}}$$

Now use the Stirling approximation

$$(n!)^n \sim (2\pi)^{n/2}n^{n^2+n/2}e^{-n^2+1/12}$$

Hence we deduce that

$$\lim_{n\to\infty} \frac{(2\pi)^{n/2}n^{n^2+n/2}e^{-n^2+1/12}}{H(n)} \times \frac{1}{(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}}$$

By simplifications we have

$$e^{1/12} \lim_{n\to\infty} \frac{n^{n^2/2+n/2+1/12}e^{-n^2/4}}{H(n)} = \frac{e^{1/12}}{A}$$

20.8 Example

Prove that

$$\zeta'(2) = \frac{\pi^2}{6}\left(\log(2\pi) + \gamma - 12\log A\right)$$

We already proved that

$$\log\left[\frac{G(n+1)}{(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}}\right] \sim \frac{1}{12} - \log A$$

Let the following

$$f(n) = \log\left[\frac{G(n+1)}{(2\pi)^{n/2} n^{n^2/2-1/12} e^{-3n^2/4}}\right]$$

Use the series representation of the Barnes functions

$$f(n) = \log\left[\frac{(2\pi)^{n/2}\exp\left(-\frac{n+n^2(1+\gamma)}{2}\right)\prod_{k=1}^{\infty}\left\{\left(1+\frac{n}{k}\right)^k \exp\left(\frac{n^2}{2k}-n\right)\right\}}{(2\pi)^{n/2} n^{n^2/2-1/12} e^{-3n^2/4}}\right]$$

Which reduces to

$$f(n) = -\frac{n+n^2(1+\gamma)}{2} + \sum_{k=1}^{\infty}\left\{k\log\left(1+\frac{n}{k}\right)+\frac{n^2}{2k}-n\right\} - \left(\frac{n^2}{2}-\frac{1}{12}\right)\log(n) + \frac{3n^2}{4}$$

Differentiate with respect to n

$$f'(n) = -\frac{1}{2} - n - \gamma n + n\psi(n) + \gamma n + 1 - n\log(n) - \frac{n}{2} + \frac{1}{12n} + \frac{3n}{2}$$

Note that we already showed that

$$\frac{d}{dn}\sum_{k=1}^{\infty}\left\{k\log\left(1+\frac{n}{k}\right)+\frac{n^2}{2k}-n\right\} = n\psi(n)+\gamma n+1$$

By simplifications we have

$$f'(n) = n\psi(n) - n\log(n) + \frac{1}{12n} + \frac{1}{2}$$

Now use that

$$\psi(n) = \log(n) - \frac{1}{2n} - 2\int_0^{\infty}\frac{zdz}{(n^2+z^2)(e^{2\pi z}-1)}dz$$

Hence we deduce that

$$f'(n) = -2\int_0^{\infty}\frac{nzdz}{(n^2+z^2)(e^{2\pi z}-1)}dz + \frac{1}{12n}$$

Integrate with respect to n

$$f(n) = -\int_0^{\infty}\frac{z\log(n^2+z^2)}{(e^{2\pi z}-1)}dz + \frac{1}{12}\log(n) + C$$

Take the limit $n \to 0$

$$C = \lim_{n\to 0} f(n) - \frac{1}{12}\log(n) + \int_0^{\infty}\frac{z\log(z^2)}{(e^{2\pi z}-1)}dz$$

Hence we have the limit

$$\lim_{n\to 0} \frac{G(n+1)}{(2\pi)^{n/2} n^{n^2/2-1/12} e^{-3n^2/4}} - \frac{1}{12}\log(n) = \lim_{n\to 0} \log \frac{G(n+1)}{(2\pi)^{n/2} n^{n^2/2} e^{-3n^2/4}} = 0$$

Hence we see that

$$C = 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)} dz$$

Finally we have

$$f(n) = -\int_0^\infty \frac{z\log(n^2+z^2)}{(e^{2\pi z}-1)} dz + \frac{1}{12}\log(n) + 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)} dz$$

$$f(n) = -\int_0^\infty \frac{z\log\left(1+\frac{z^2}{n^2}\right)}{(e^{2\pi z}-1)} dz - \log(n^2)\int_0^\infty \frac{z}{(e^{2\pi z}-1)} dz + \frac{1}{12}\log(n) + 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)} dz$$

Also we have

$$\int_0^\infty \frac{z}{(e^{2\pi z}-1)} dz = \frac{1}{24}$$

That simplifies to

$$f(n) = -\int_0^\infty \frac{z\log\left(1+\frac{z^2}{n^2}\right)}{(e^{2\pi z}-1)} dz + 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)} dz$$

Take the limit $n \to \infty$

$$2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)} dz = \frac{1}{12} - \log A$$

Now use that

$$\begin{aligned}
2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)} dz &= 2\int_0^\infty \frac{z\log(z)}{e^{2\pi z}} \times \frac{1}{1-e^{-2\pi z}} dz \\
&= 2\sum_{n=0}^\infty \int_0^\infty e^{-2\pi z(n+1)} z\log(z)\, dz \\
&= \sum_{n=1}^\infty \frac{\psi(2) - \log(2\pi) + \log(n)}{2\pi^2 n^2} \\
&= \frac{(\psi(2) - \log(2\pi))\zeta(2) + \zeta'(2)}{2\pi^2}
\end{aligned}$$

Hence we conclude that

$$\zeta'(2) = (\log(2\pi) - \psi(2))\zeta(2) + 2\pi^2\left(\frac{1}{12} - \log A\right) = \zeta(2)(\log(2\pi) + \gamma - 12\log A)$$

20.9 Example

Prove that

$$\zeta'(-1) = \frac{1}{12} - \log A$$

proof

Start by

$$\zeta(s) = \lim_{m\to\infty}\left(\sum_{k=1}^{m} k^{-s} - \frac{m^{1-s}}{1-s} - \frac{m^{-s}}{2} + \frac{sm^{-s-1}}{12}\right)\;,\;\mathrm{Re}(s) > -3.$$

Differentiate with respect to s

$$\zeta'(s) = \lim_{m\to\infty}\left(-\sum_{k=1}^{m} k^{-s}\log(k) - \frac{m^{1-s}}{(1-s)^2} + \frac{m^{1-s}}{1-s}\log(m) + \frac{m^{-s}}{2}\log(m) + \frac{m^{-s-1}}{12} - \frac{m^{-s-1}}{12}\log(m)\right)$$

Now let $s \to -1$

$$\zeta'(-1) = \lim_{m\to\infty}\left(-\sum_{k=1}^{m} k\log(k) - \frac{m^2}{4} + \frac{m^2}{2}\log(m) + \frac{m}{2}\log(m) + \frac{1}{12} - \frac{1}{12}\log(m)\right)$$

Take the exponential of both sides

$$e^{\zeta'(-1)} = e^{1/12}\lim_{m\to\infty}\frac{m^{m^2/2+m/2-1/12}e^{-m^2/4}}{e^{\sum_{k=1}^{m} k\log(k)}} = e^{1/12}\lim_{m\to\infty}\frac{m^{m^2/2+m/2-1/12}e^{-m^2/4}}{H(m)} = \frac{e^{1/12}}{A}$$

We conclude that

$$\zeta'(-1) = \frac{1}{12} - \log A$$

20.10 Relation to Howrtiz zeta function

Prove that

$$\log G(z+1) - z\log\Gamma(z) = \zeta'(-1) - \zeta'(-1,z)$$

proof

Start by the following

$$\zeta(s,z) = \frac{z^{-s}}{2} + \frac{z^{1-s}}{s-1} + 2\int_0^\infty \frac{\sin(s\arctan(x/z))}{(z^2+x^2)^{s/2}(e^{2\pi x}-1)}\,dx$$

Take the derivative with respect to s and $s \to -1$

$$\zeta'(-1,z) = -\frac{z\log(z)}{2} + \frac{z^2\log(z)}{2} - \frac{z^2}{4} + \int_0^\infty \frac{x\log(x^2+z^2) + 2z\arctan(x/z)}{(e^{2\pi x}-1)}\,dx$$

Now use that

$$\psi(z) = \log(z) - \frac{1}{2z} - 2\int_0^\infty \frac{x}{(z^2+x^2)(e^{2\pi x}-1)}dx$$

Which implies that

$$\int_0^\infty \frac{2zx}{(z^2+x^2)(e^{2\pi x}-1)}dx = z\log(z) - \frac{1}{2} - z\psi(z)$$

By taking the integral

$$\int_0^\infty \frac{x\log(x^2+z^2) - x\log(x^2)}{(e^{2\pi x}-1)}dx = \int_0^z x\log(x)\,dx - \int_0^z x\psi(x)\,dx - \frac{z}{2}$$

Which simplifies to

$$\int_0^\infty \frac{x\log(x^2+z^2)}{(e^{2\pi x}-1)}dx = \zeta'(-1) - \frac{z^2}{4} + \frac{1}{2}z^2\log(z) - z\log\Gamma(z) + \int_0^z \log\Gamma(x)\,dx - \frac{z}{2}$$

Also we have

$$2\int_0^\infty \frac{x}{(x^2+z^2)(e^{2\pi x}-1)}dx = \log(z) - \frac{1}{2z} - \psi(z)$$

By integration we have

$$2\int_0^\infty \frac{\arctan(x/z)}{(e^{2\pi x}-1)}dx = z + \frac{\log(z)}{2} - z\log(z) + \log\Gamma(z) + C$$

Let $z \to 1$ to evaluate the constant

$$2\int_0^\infty \frac{\arctan(x/z)}{(e^{2\pi x}-1)}dx = z + \frac{\log(z)}{2} - z\log(z) + \log\Gamma(z) - \frac{1}{2}\log(2\pi)$$

Multiply by z

$$2\int_0^\infty \frac{z\arctan(x/z)}{(e^{2\pi x}-1)}dx = z^2 + \frac{z\log(z)}{2} - z^2\log(z) + z\log\Gamma(z) - \frac{z}{2}\log(2\pi)$$

Substitute both integrals in our formula

$$\int_0^z \log\Gamma(x)\,\mathrm{d}x = \frac{z}{2}\log(2\pi) + \frac{z(1-z)}{2} - \zeta^{'}(-1) + \zeta^{'}(-1,z)$$

We also showed that

$$\int_0^z \log\Gamma(x)\,\mathrm{d}x = \frac{z}{2}\log(2\pi) + \frac{z(1-z)}{2} + z\log\Gamma(z) - \log G(z+1)$$

By equating the equations we get our result.

20.11 Example

Prove that

$$G\left(\frac{1}{2}\right) = 2^{1/24}\pi^{-1/4}e^{1/8}A^{-3/2}$$

proof

We know that

$$\log G(z) + \log\Gamma(z) - z\log\Gamma(z) = \zeta'(-1) - \zeta'(-1,z)$$

Note that

$$\zeta\left(s,\frac{1}{2}\right) = (2^s - 1)\zeta(s)$$

Which implies that

$$\zeta'\left(-1,\frac{1}{2}\right) = \frac{\log(2)}{2}\zeta(-1) - \frac{1}{2}\zeta'(-1)$$

Hence we have

$$\log G\left(\frac{1}{2}\right) + \frac{1}{2}\log\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\zeta'(-1) - \frac{\log(2)}{2}\zeta(-1)$$

Using that we have

$$G\left(\frac{1}{2}\right) = 2^{1/24}\pi^{-1/4}e^{\frac{3}{2}\zeta'(-1)}$$

Note that

$$\zeta(-1) = -\frac{1}{12}$$

This can be proved by the functional equation of the zeta function.

21 Complex Analysis

21.1 Introduction to complex numbers

The idea of complex numbers originated from trying to solve polynomials like those in the form $x^2+1=0$. If we do a simple algebra then it is easy to induce that $x^2=-1$ but we know that there is no real number whose square is a negative number. Which implies that this polynomial has no solutions in the set of real numbers. Hence, we need to expand the set of real numbers $\mathbb{R}$ to the set of complex numbers $\mathbb{C}$. By definition a complex number can be written in the form $z=x+iy$ where $x,y\in\mathbb{R}$. We say the real part of $\Re(z)=x$ and the imaginary part $\Im(z)=y$. We can think of i as a special symbol with the property $i^2=-1$ or $i=\sqrt{-1}$. Now let us define some algebraic operations.

Let $a=x_1+iy_1$ and $b=x_2+iy_2$ be two complex numbers then we define the following operations

$$a\pm b=(x_1+x_2)\pm i(y_1+y_2)$$

$$a\times b=x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)$$

For division we need that $b\neq 0$ which by definition means both x_2 and y_2 can't be zero at the same time. Also we need to define the complex conjugate which is $\bar{a}=x_1-iy_1$ then we can easily see that $a\times\bar{a}=x_1^2+y_1^2$. Using that we define division for a and $b\neq 0$ as

$$\frac{a}{b}=\frac{a}{b}\times\frac{\bar{b}}{\bar{b}}=\frac{a\bar{b}}{x_2^2+y_2^2}$$

21.2 Polar representation

We usually define complex numbers as the following

$$z=re^{i\phi}=r(\cos(\phi)+i\sin(\phi))$$

Where we define $r=|z|=\sqrt{x^2+y^2}$ as the modulus and ϕ called the argument or $\arg(z)$. Geometrically this can be seen as in Figure 1.

This representation allows a better representation of both multiplication and division. Let $z_1=r_1e^{i\phi_1}$ and $z_2=r_2e^{i\phi_2}$ be two complex numbers. Then this implies

$$z_1\times z_2=r_1r_2e^{i(\phi_1+\phi_2)}$$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}e^{i(\phi_1-\phi_2)}$$

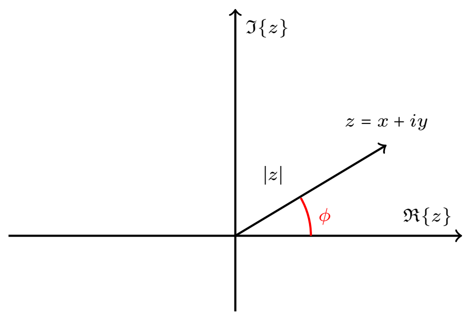


Figure 1: Polar representation of a complex number

The absolute value of complex number or the length of the vector representation is an important property of a complex number. Note that $|z| = 1$ implies that z is unit vector. One of the most important properties is the triangle inequality which will be very handy for us in the coming sections

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

To prove it note that

$$\begin{aligned}|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} &= |z_1|^2 + z_1\bar{z_2} + z_2\bar{z_1} + |z_2|^2 \\ &= |z_1|^2 + z_1\bar{z_2} + \overline{z_1\bar{z_2}} + |z_2|^2 \\ &= |z_1|^2 + 2\mathrm{Re}(z_1\bar{z_2}) + |z_2|^2\end{aligned}$$

Note also

$$\mathrm{Re}(z_1\bar{z_2}) \leq |z_1\bar{z_2}| = |z_1||z_2|$$

Finally we have

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1z_2| + |z_2|^2 = (|z_1| + |z_2|)^2$$

An easy corollary is the reverse triangle inequality

$$|z_1 + z_2| \geq |z_1| - |z_2|$$

One can also deduce the following by induction on n

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$$

Note that geometrically in terms of vectors the triangle inequality implies that the length of two vectors is greater than the length of their sum. This also holds in triangles in Euclidean geometry where the sum of lengths of two sides is greater than the third the length.

21.3 Complex functions

Complex functions are mapping $f : \mathbb{C} \to \mathbb{C}$. A function that is differentiable on a certain neighborhood of the plain is called an analytic function. You can see a specialized complex analysis book to understand these terms in details. Specifically the relation to Cauchy-Riemann equations.

21.3.1 Exponential function

The exponential function $f = \exp(z)$ is an entire function (analytic/differentiable in the whole plain). We see that the function has no discontinuities. Also we have

$$e^z = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$$

Which implies that $|e^z| = e^x$. The function acts smoothly with the usual derivative $(e^z)' = e^z$ and the expansion

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

One also can note that the function is periodic with period $2k\pi i$.

21.3.2 Sine and Cosine and hyperbolic functions

These functions are also entire in the complex plane. They almost have the same properties as the real counterparts but one essential difference is boundedness. We know in the real case that $|\sin(x)| \leq 1$ but it is not the case for $|\sin(z)|$. You can see that since $\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$. Now since $|e^{-iz}| = e^y$ we see that e^y is unbounded near infinity then $\sin(z)$ is also unbounded. You can deduce the same for $\cos(z)$. Note that also $\frac{d}{dz}\sin(z) = \cos(z)$ also we have $\frac{d}{dz}\cos(z) = -\sin(z)$. The expansions around zero are

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2k+1)!} z^{2k+1}$$

$$\cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^n}{(2k)!} z^{2k}$$

The hyperbolic functions can be defined in terms of the sine and cosine functions using the relations

$$\cosh(z) = \cos(iz)\ ,\ \sinh(z) = -i\sin(iz)$$

You can then deduce the derivatives and the series expansions around zero. The exponential representation is also useful. Note that

$$\cosh(z) = \frac{e^z + e^{-z}}{2}\ ,\ \sinh(z) = \frac{e^z - e^{-z}}{2}$$

21.3.3 Complex logarithm

Usually we think of logarithm as the inverse of the exponential function away from zero. As in real analysis we have $\log(e^x) = x$ but it is completely risky to think the same for complex analysis. The problem with complex exponential is its periodicity. Note that we have $e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z$, hence the function is not one to one. We have infinite values that map to the same value. This makes the complex logarithm a multi-valued function with the definition

$$\log(z) = \log|z| + i\arg(z)$$

Some authors might use $\ln(z)$ interchangeably with $\log(z)$. Here we usually use $\log(z)$ to denote the natural complex logarithm. Let us see some examples

Example. Find $\log(-2)$, $\log(i)$ and $\log(1+i)$

solution

The easiest way is to use the polar representation. Note that we have $e^{i\pi} = -1$ which implies $2e^{i\pi+2k\pi i} = -2$. Simply we have

$$\log(-2) = \log(2) + i(\pi + 2k\pi)$$

Similarly we have

$$\log(i) = \log(1) + i\left(\frac{\pi}{2} + 2k\pi\right) = i\left(\frac{\pi}{2} + 2k\pi\right)$$

Finally since $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ which implies

$$\log(1+i) = \log(\sqrt{2}) + i\left(\frac{\pi}{4} + 2k\pi\right)$$

Properties

1. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

2. $\log(z^n) = n\log(z)$

3. $e^{\log(z)} = z$

proof

1. Suppose that $z_1 = r_1 e^{i\phi_1}$, $z_2 = r_2 e^{i\phi_2}$ and $m = k + n$

$$\begin{aligned}\log(z_1 z_2) + 2n\pi i &= \log(r_1 r_2 e^{i(\phi_1+\phi_2)}) \\ &= \log(r_1 r_2) + i\,(\phi_1 + \phi_2 + 2m\pi) \\ &= \log(r_1) + i(\pi_1 + 2n\pi) + \log(r_2) + i(\phi_2 + 2k\pi) \\ &= \log(z_1) + \log(z_2)\end{aligned}$$

2. Similarly we have for $z = re^{i\phi}$

$$\log(z^n) = \log(r^n e^{in\phi}) = n\log(r) + i(n\phi + 2kn\pi) = n\log(z)$$

3. Finally we have

$$e^{\log(z)} = e^{\log|r| + i(\phi+2k\pi)} = re^{i\phi} = z$$

Some properties don't necessarily hold for example $\log(e^z) \neq z$ since $e^{z+2k\pi i} = e^z$ and that implies $z = z + 2k\pi i$ which is obviously can't be true unless we choose $k = 0$. This suggests that we can make nice properties by choosing proper values of the argument. This raises the concept of **Branches of logarithm**. The definition $\log(z) = \log|z| + i(\phi + 2k\pi)$ makes the complex logarithm multi valued as explained earlier. Each value for k raises a different branch of the logarithm that makes it single valued. One interesting branch is called **Principal logarithm**. We define it as the following

$$\text{Log}(z) = \log|z| + i\text{Arg}(z)$$

where we define $-\pi < \text{Arg}(z) \leq \pi$. Note that we usually use $\log(z)$ to denote the principal value when it is clear from the context. Clearly this makes the function single valued.

Example. Find $\text{Log}(-2)$ and $\text{Log}(1+i)$

solution

The main trick it to find the value of the argument that falls in the interval $(-\pi, \pi]$. Using that we conclude

$$\log(-2) = \log(2) + i\pi$$

Similarly we have

$$\log(1+i) = \log(\sqrt{2}) + i\frac{\pi}{4}$$

The usual properties described in the previous section don't usually carry on for the principal value. For instance, $\text{Log}(z_1 z_2) \neq \text{Log}(z_1) + \text{Log}(z_2)$. Let $z_1 = z_2 = -1$ and note that

$$\text{Log}(-1 \times -1) = \log(1) = 0 \neq \text{Log}(-1) + \text{Log}(-1) = 2\text{Log}(-1) = 2\pi i$$

Now let us define the concept of a **Branch cut**. In order to make the complex logarithm an analytic function we have to make it differentiable in a certain neighborhood. Note that the principal value of the logarithm by choosing a certain branch the function became one-to-one but this is not enough because the function $\text{Log}(z)$ doesn't behave well on the negative real axis.

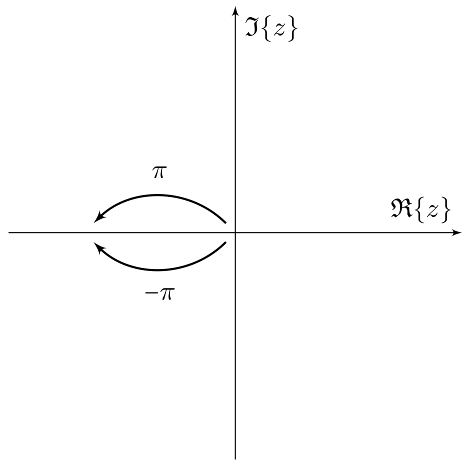


Figure 2: Behaviour of the Principal Logarithm near the branch cut

We see from the graph in Figure 2 that the function approach different value as we approach the negative real axis. This causes a discontinuity that prevents the function from being analytic on that line.

Theorem 1 *The function with* $|z| \neq 0$

$$\text{Log}(z) = \log|z| + i\phi \ , \ \phi \in (-\pi, \pi)$$

is analytic and the derivative is $\frac{1}{z}$.

Using that we can expand the function along values away from the branch cut. For instance for $|z| < 1$ we have

$$\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}$$

We are not going to go into details of the proofs of such claims because they require a more firm understanding of analytic functions.

21.4 Taylor and Laurent expansions

In this section we establish the foundations of contour integration by studying the theory of Laurent expansion, residues and poles. It is preferable if you have a basic knowledge in real analysis because many theorems carry on from the real case to the complex plane. We start by the following theorem

Theorem 2 *Let f be an analytic function in a domain D and z_0 be a point in D Then*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

is a valid representation in the largest circle with radius $|z - z_0|$ contained in D.

Example. Find the expansion of $\frac{1}{z}$ around $z_0 = 2$.

solution

Since the function is analytic in the whole plane except at the origin. Then we can use the Theorem to deduce that

$$f^{(k)}(2) = \frac{(-1)^k k!}{2^{k+1}}$$

You can verify that by taking multiple derivatives and substitute the value of $z_0 = 2$. Hence we deduce that

$$\frac{1}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k k!}{2^{k+1}}(z - 2)^k$$

And the radius of convergence is $|z - 2| < 2$ which represents a ball centered at $z_0 = 2$ with radius less than 2. As we see from the Figure 3.

Note that we cannot include zero because the function is not defined there. So the maximum circle is that of radius 2.

Theorem 3 *Let f be an analytic function in the punctured disk $r < |z - z_0| < R$. Then f has a series representation*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

The theorem is actually saying that if a function is not analytic on a point z_0 but analytic around it then we can expand the function using what we call a **Laurent expansion**. The difference between

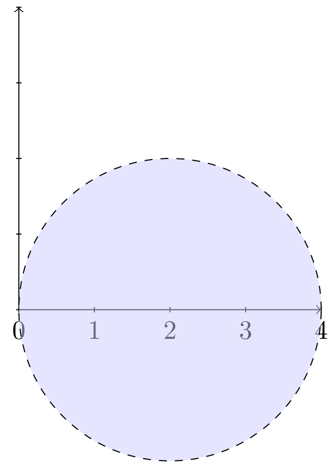


Figure 3: Radius of convergence

Laurent expansion and Taylor is that Laurent allows negative indices of the series. We can think of it as a generalization of the Taylor theorem which is a special case where coefficients with negative indices are zero.

Example Expand the function $f(z) = \frac{1}{z(1-z)}$ in the domain $0 < |z| < 1$.

solution

We need to rewrite the function as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

We need to expand both functions in the domain $0 < |z| < 1$. Note that the function $f(z) = \frac{1}{z}$ is a Laurent expansion around zero in the punctured disk $|z| > 0$ what is remaining is to expand the function $f(z) = \frac{1}{1-z}$ in the domain $|z| < 1$. Hence the intersection of both domains is actually $0 < |z| < 1$. Note that

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$$

is a Taylor expansion valid in $|z| < 1$ hence we have

$$f(z) = \frac{1}{z} \sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} z^{k-1} = \sum_{k=-1}^{\infty} z^k$$

which is exactly what we want.

Example Find the Laurent expansion of the function $f(z) = \frac{\sin(z)}{z^2}$ around $z = 0$.

solution

Note that

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$$

Which is a valid expansion in the domain $z \in \mathbb{C}$. Also $\frac{1}{z^2}$ is a valid expansion in the domain $|z| > 0$. Hence

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k-1}$$

is a Laurent expansion in the domain $|z| > 0$.

Example. Find the Laurent expansion of the function $f(z) = \frac{e^z}{z-1}$ around $z = 1$.

solution

First note that around $z = 0$ we have

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

Now let $z \to z - 1$, then

$$e^{z-1} = \sum_{k=0}^{\infty} \frac{1}{k!}(z-1)^k$$

Which implies that

$$e^z = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!}(z-1)^k$$

is a valid expansion in the whole domain $\mathbb{C}$. Finally we have

$$f(z) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!}(z-1)^{k-1}$$

in the punctured disk $|z - 1| > 0$.

21.5 Poles and residues

Here we discuss the of types of points where a complex function is not defined at. Usually they are called singularities. Mainly we start by discussing the concept of **Isolated Singularities**.

An isolated singular point is a point where the function is analytic in the punctured disk by removing that point.

1. **Removable singularity** A singular point is removable if $\lim_{z \to z_0} f(z)$ exists. Equivalently, if all the negative indexed terms in the Laurent expansion are zeros,namely $a_{-k} = 0$ for all $k > 0$.

2. **Poles** a function f has a pole of order $m > 0$ if $a_{-m} \neq 0$ and $a_{-k} = 0$ for all $k > m$.

3. **Essential singularity** The Laurent expansion has an infinite number of non-positive indexed terms.

Example. Classify the following singularities

1. $f(z) = \frac{\sin(z)}{z}$ at $z = 0$

2. $f(z) = \frac{e^z}{(z-1)^2}$ at $z = 1$

3. $f(z) = e^{1/z}$ at $z = 0$

solution

1. This is a removable singularity

$$\lim_{z \to 0} \frac{\sin(z)}{z} = 1$$

2. By expanding around 1 we have

$$f(z) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{1}{k!} (z-1)^{k-2}$$

This implies 1 is a pole of order 2.

3. Note that

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k! z^k}$$

Then $z = 0$ is an essential singularity.

Theorem 4 *The function f has a pole of order m at z_0 if and only if*

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

where $g(z_0) \neq 0$.

The proof is done by writing the Laurent expansion of f and take $(z-z_0)^{-m}$ as a common factor.

Residues are closely related to the Laurent expansion of a function. They will become really handy when we start discussion about contours. We usually use the following notation $\mathrm{Res}(f, z_0)$ as the residue of the function $f(z)$ at $z = z_0$.

Definition. Let $f(z)$ be an analytic function in a domain D with the Laurent expansion

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

around $z = z_0$. Then we define

$$\mathrm{Res}(f, z_0) = a_{-1}$$

Where a_{-1} is the cofficient of $(z - z_0)^{-1}$

Example. Find the residues of the following functions around $z = 0$

1. $f(z) = \frac{\sin(z)}{z}$
2. $f(z) = \frac{e^z}{z^2}$
3. $f(z) = e^{1/z}$

solution

1. Note that since f has a removable singularity and since $a_{-k} = 0$ for all $k > 0$ then we have $\mathrm{Res}(f, 0) = 0$.

2. It is simple to see that

$$f(z) = \frac{1}{e} \sum_{k=-2}^{\infty} \frac{1}{(k+2)!} z^k$$

at $k = -1$ we conclude

$$\mathrm{Res}(f, 0) = \frac{1}{e}$$

3. Using the Taylor expansion of e^z and letting $z \to 1/z$ and rewrite the expansion as

,

$$e^{1/z} = \sum_{k=-\infty}^{0} \frac{1}{(-k)!} z^k$$

Hence we have

$$\mathrm{Res}(f, 0) = 1$$

Usually we don't have to find the Laurent expansion in order to find the residues at a specific point. Let us first note that the residues of an entire function is zero at any point. Now let us see theorems that help us find the residues much more easier

Theorem 5 *Suppose z_0 is a pole of order m of $f(z)$ then*

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z)$$

proof

Write the Laurent expansion of f

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{(z - z_0)} + \cdots$$

Multiply by $(z - z_0)^m$

$$(z - z_0)^m f(z) = a_{-m} + \cdots + a_{-1}(z - z_0)^{m-1} + \cdots$$

Take the derivative of both sides

$$\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) = (m-1)! a_{-1} + O((z - z_0))$$

We finish by Letting $z \to z_0$

$$\frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_0)^m f(z) = a_{-1}$$

Theorem 6 *Suppose we have the following function*

$$h(z) = \frac{f(z)}{g(z)}$$

where z_0 is a simple zero of $g(z)$ and $f(z_0) \neq 0$ then

$$\operatorname{Res}(h, z_0) = \frac{f(z_0)}{g'(z_0)}$$

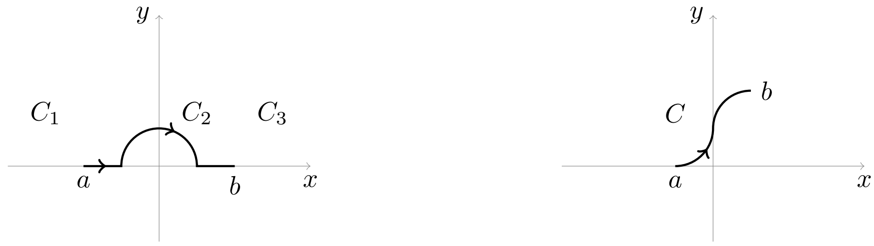
proof

Since z_0 is a simple pole of h then

$$\operatorname{Res}(h, z_0) = \lim_{z \to z_0} (z - z_0) \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z)}{\frac{g(z) - g(z_0)}{z - z_0}} = \frac{f(z_0)}{g'(z_0)}$$

21.6 Integration around paths

Since in the complex plane $\mathbb{C}$ every complex number $z \in \mathbb{C}$ has two components real and complex. We can not use the same process as in real analysis to evaluate integrals. Now, integration happens around paths. We will be concerned about a family paths. Mainly those curves that have continuous derivatives. These are called **smoothed curves**. Mainly these curves don't have a peak where the derivative doesn't exist, hence named smooth. We can attach a finite number of smooth curves to obtain **piece-wise smooth curves**. Also those curves must be **simple**, that means it doesn't cross itself.



(a) Piece-wise smooth curve

(b) Smooth curve

Figure 4: Examples of smooth curves

Note that the orientation of a curve is essential to the evaluation of a path integral. Usually we work with anti-clockwise orientations.

Theorem 7 *Let f be continuous on a smooth curve γ given by $\gamma(t) = x(t) + iy(t)$, where $t \in [a,b]$ then*

$$\int_{\gamma} f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Example

Integrate the function $f(z) = z^2$ around a circle of radius 1 traced counter-clockwise.

solution

Denote the curve $\gamma(t) = e^{it}$ where $t \in [0, 2\pi)$. Now since z^2 is continuous on the simple smooth path we have

$$\int_{\gamma} z^2 dz = i \int_0^{2\pi} e^{2it} e^{it} dt = \frac{1}{3}\left(e^{2\pi i} - e^0\right) = 0$$

This seems to imply that if we integrate around a simple closed curve the integration will be always zero. This is not always the case, consider integrating z^{-1} around the same curve

$$\int_{\gamma} z^{-1} dz = i \int_0^{2\pi} e^{-it} e^{it} dt = i \int_0^{2\pi} dt = 2\pi i$$

One might notice that the function $f(z) = z^{-1}$ has a simple pole inside the closed path while $f(z) = z^2$ is analytic in and on the contour. This is indeed the case as the Cauchy's theorem illustrates. We first define **simply connected domains**. A domain is simply connected if every closed contour/path can be shrunk into one point. That implies the domain can't contain holes. Let us know take a look at the **Cauchy-Grousat Theorem.**

Theorem 8 *Let f is an analytic in a simpley connected domain D. Then for every simple closed contour C in D we have*

$$\oint_C f(z)dz = 0$$

Example. Choose a closed simple contour $\gamma(t)$ in the complex plain then find the integration of $f(z) = (z^2 + 3z + 1)^n$ where $n \geq 1$ around that contour.

solution

We don't have to worry about parametrizing the integral since the function is entire. We apply the theorem directly to conclude that

$$\int_\gamma f(z) = \int_C (z^2 + 3z + 1)^n = 0$$

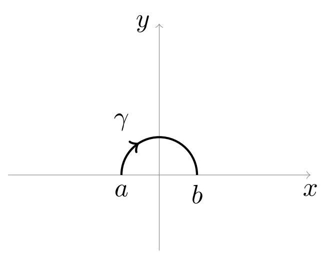
One might think that regardless of the path of integration we can deduce that **Independence of paths** which is explained in the theorem.

Theorem 9 *For any analytic function f in a connected domain D for any simple paths γ_1, γ_2 inside D we have*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

Where γ_1, γ_2 have the same initial and final points.

Example Evaluate the function $f(z) = z$ around the following path



By independence of paths we can use a simple contour (line) that connect the same points a and b. Define $\gamma(t) = bt + (1-t)a$ where $t \in [0, 1]$.

$$\int_\gamma z dz = (b-a)\int_0^1 bt + (1-t)a\, dt = b^2 - a^2 - \frac{b^2-a^2}{2} = \frac{b^2-a^2}{2}$$

One can also use the anti-derivative

$$\int_\gamma z dz = \left[\frac{z^2}{2}\right]_a^b = \frac{b^2-a^2}{2}$$

This is indeed true by the fundamental theorem.

Theorem 10 *For any analytic function f in a connected domain D for any simple paths $\gamma(t), t \in [a,b]$ inside D we have*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

So if the function is analytic and by the independence of path theorem we can find the anti-derivative and use the initial and final points.

Now we look at the approach to evaluate functions that have poles inside the contour. First let us look at the most important theorem , **Cauchy's Integral formula**.

Theorem 11 *Let f be an analytic function in a simply connected domain D then for any contour $\gamma \in D$ we have*

$$\oint_\gamma \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)$$

proof

First we can deform the contour γ into a circular contour C. Then we have the following

$$\int_\gamma \frac{f(z)}{z-z_0}dz = \int_C \frac{f(z)}{z-z_0} = f(z_0)\int_C \frac{dz}{z-z_0} + \int_C \frac{f(z)-f(z_0)}{z-z_0}dz$$

Note the second function is analytic since it has a removable singularity at $z = z_0$ hence

$$\int_C \frac{f(z)-f(z_0)}{z-z_0}dz = 0$$

Hence we conclude

$$\int_\gamma \frac{f(z)}{z-z_0}dz = f(z_0)\int_C \frac{dz}{z-z_0} = 2\pi i\, f(z_0)$$

Using that we can conclude the **Cauchy's Integral formula for Derivatives**

$$\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

This theorem implies that the evaluation of a closed contour depends on the residue evaluation. Since every analytic function f can be written as

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

Then we have

$$\frac{f(z)}{(z - z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{(k!)(z - z_0)} + \sum_{k \neq n}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

So the only term that contributes to the integral is the coefficient of $(z - z_0)^{-1}$ which is by definition the residue. Hence we conclude that if f has a singularity $z = z_0$ we have

$$\oint_\gamma f(z)dz = 2\pi i \operatorname{Res}(f, z_0)$$

We generalize the theorem by the following theorem **Cauchy Residue Theorem** or the **residue theorem**.

Theorem 12 *Let f be an analytic function on a connected domain D except for some isolated points $z_1, \cdots, z_n$ then for any simple closed contour $\gamma \subset D$ that contains the singularities we have*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

proof

Deform the contour γ into a set of circular contours $\gamma_1, \cdots, \gamma_n$ around each singularity

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

Note for any $1 \leq k \leq n$

$$\int_{\gamma_k} f(z)dz = 2\pi i \operatorname{Res}(f, z_k)$$

Hence we deduce the result as sum of the residues. This is very powerful since it explains that the evaluation of a function with some singularity around a contour containing them is no more than the sum of residues. This relationship will be essential when we try to evaluate real integrals.

Example. Let $P(z)$ be a polynomial with the leading coefficient as 1. Then find the following

$$\int_\gamma \frac{dz}{P(z)}$$

where γ contains the zeros of P.

solution

Factor $P(z) = (z - z_0)(z - z_1)$. Consider two cases

Case 1. the polynomial has a repeated root z_0 then

$$\int_\gamma \frac{dz}{P(z)} = \int_\gamma \frac{dz}{(z - z_0)^2} = 2\pi i \text{Res}\left(\frac{1}{(z - z_0)^2}, z_0\right) = 0$$

Since the Laurent expansion doesn't contain a coefficient of $(z - z_0)^{-1}$.

Case 2. the polynomial has a two distinctive roots z_0, z_1 then

$$\int_\gamma \frac{dz}{P(z)} = \int_\gamma \frac{dz}{(z - z_0)(z - z_1)} = \sum_{k=0}^{1} \text{Res}(1/P(z), z_k)$$

Note that

$$\text{Res}(1/P(z), z_0) = \lim_{z \to z_0} \frac{(z - z_0)}{(z - z_0)(z - z_1)} = \frac{1}{z_0 - z_1}$$

$$\text{Res}(1/P(z), z_1) = \lim_{z \to z_1} \frac{(z - z_1)}{(z - z_0)(z - z_1)} = \frac{1}{z_1 - z_0}$$

Hence we conclude that also

$$\int_\gamma \frac{dz}{P(z)} = 0$$

Can you generalize that ?

21.7 Bounds on integrals

Some integrals on contours are too difficult to evaluate so we better find a good bound or prove the integral goes to 0 by proving

$$\left| \oint_\gamma f(z) dz \right| \leq \epsilon$$

So the integral can be made arbitrary close hence it goes to 0. We start by an important bound called the **Estimation lemma**.

Theorem 13 *Let f be a complex-valued, continuous function on the contour c if f is bounded by M for all z on c then*

$$\int_c f(z) dz \leq ML$$

where L is the length of the curve c.

proof

Note that by definition we have

$$\left|\int_c f(z)\,dz\right| \le \int_c |f(z)|\cdot|dz|$$

Now assume that $|f(z)| \le M$ that means the function is bounded on the curve

$$\int_c |f(z)|\cdot|dz| \le M\int_c |dz|$$

Now assume that the $c = \gamma(t)$ is a parameterization of the curve then

$$\int_c |dz| = \int_a^b |\gamma'(t)|\,dt$$

Now by the definition of the length L of a curve we have

$$\int_a^b |\gamma'(t)|\,dt = L$$

Example. Prove that

$$\lim_{R\to\infty}\int_{C_R}\frac{1}{z^2+1}dz = 0$$

Where C_R is a semi-circle of radius R and centered at the origin.

proof

First note that for any point on the curve we have $|z| = R$ hence

$$\left|\frac{1}{z^2+1}\right| \le \frac{1}{R^2-1}$$

Where we used the Triangle inequality to show $|z| - 1 \ge R - 1$ for large R. Also note that length of the semi circle contour is πR. Hence we deduce

$$\left|\int_{C_R}\frac{1}{z^2+1}dz\right| \le \frac{\pi R}{R^2-1}$$

By taking $R \to \infty$ we conclude our result.

The estimation lemma could be used to prove a more generalized form which we call the **Jordan's lemma**.

Theorem 14 *Let f be a complex-valued, continuous function on the contour C_R which is a semi-circle on the upper half plane centered at the origin. Let f be defined as the following*

$$f(z) = e^{iaz} g(z)$$

For some $a > 0$ and assume that $|f| \leq M_R$ on C_R then

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{a} M_R$$

proof

By using the bound on g we have

$$\left| \int_{C_R} f(z) dz \right| \leq M_R \int_{C_R} |e^{iaz}| \cdot |dz|$$

Now note that on the curve for $t \in (0, \pi)$

$$|e^{iaz}| = \left| e^{iaR(\cos(t) + i \sin(t))} \right| \leq e^{-aR \sin(t)}$$

Hence we deduce by the parameterization Re^{iat}

$$\left| \int_{C_R} f(z) dz \right| \leq RM_R \int_0^{\pi} e^{-aR \sin(t)} dt$$

Since $\sin(t)$ is symmetric around $\pi/2$ we have

$$\int_0^{\pi} e^{-aR \sin(t)} dt = 2 \int_0^{\pi/2} e^{-aR \sin(t)} dt$$

Now use the fact $\sin(t) \geq \frac{2t}{\pi}$ to deduce

$$2 \int_0^{\pi/2} e^{-aR \sin(t)} dt \leq 2 \int_0^{\pi/2} e^{-2atR/\pi} dt = \frac{\pi}{aR} \left(1 - e^{-aR}\right) \leq \frac{\pi}{aR}$$

We finally get the result

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{aR} \times RM_R = \frac{\pi}{a} M_R$$

Example. Prove that

$$\lim_{R \to \infty} \int_{C_R} \frac{e^{2iz}}{z^2 + 1} dz = 0$$

Where C_R is a semi-circle of radius R and centered at the origin.

proof

By Jordan's lemma

$$\left|\int_{C_R} \frac{e^{2iz}}{z^2+1}dz\right| \leq \frac{\pi}{2}\frac{1}{R^2-1}$$

It is left as an exercise to show the same result holds for

$$f(z) = e^{iaz}\frac{P(z)}{Q(z)}$$

where $P(z)$ and $Q(z)$ are polynomials and $D(P) \leq D(Q) + 1$. Another remark is that we can generalize the result to contours in the lower half-plain but wit the condition $a < 0$.

21.8 Contours around poles

Now we look at the case where we have a singular point along the path of integration. To define that we have to look at the concept **Cauchy Principal Value**. Let use first look at the definition in the real case. Consider the case where f is continuous on the interval $[a, b]$ except for c in that interval then

$$PV\int_a^b f(x)dx = \lim_{\epsilon\to 0^+}\int_a^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^b f(x)dx$$

Interestingly we can look at the point $x = c$ as a singular point of the function $f(x)$.

Example. Find the principal value of the function $f(x) = \frac{1}{x}$ on the interval $x \in (-1, 1)$.

solution

First note that the integral in the usual definition of Riemann doesn't exist because the function is not continuous in the interval. Now let us look at the principal value

$$PV\int_{-1}^1 \frac{1}{x}dx = \lim_{\epsilon\to 0^+}\int_{-1}^{-\epsilon}\frac{1}{x}dx + \int_{\epsilon}^1\frac{1}{x}dx == \lim_{\epsilon\to 0^+} -\int_{\epsilon}^1\frac{1}{x}dx + \int_{\epsilon}^1\frac{1}{x}dx = 0$$

Hence the principal value exists while the integral is infinite on the interval. But if the Riemann integral exists then the principal value integral exists and they are equal.

We can generalize the case to complex functions. Suppose that the function f around the along the contour C has a pole at the point $z = z_0$ then we can make a detour (semi-circle) around the pole and take the radius goes to 0 as in the Figure 5.

Suppose that $f(z)$ has a singularity at $z = 0$ and we need to find the integral along the contour C that is a line connecting a and b then

$$PV\int_C f(z)dz = \lim_{\epsilon\to 0}\int_a^{-\epsilon} f(x)dx + \int_{C_\epsilon} f(z)dx + \int_{\epsilon}^b f(x)dx$$

Let us now look at a simpler way to evaluate the contour around a simple pole instead of taking the limit.

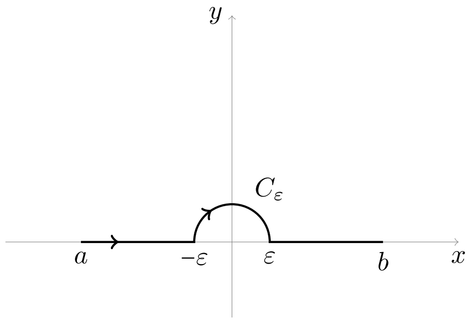


Figure 5: Avoiding singular points

Theorem 15 *Let $f(z)$ have a simple pole at $z = z_0$. Suppose that C_r is an arc starting at an angle θ_i and ending at an angle θ_f around z_0 then*

$$\lim_{r\to 0}\int_{C_r}\frac{f(z)}{z-z_0}dz = i(\theta_f - \theta_i)f(z_0)$$

This theorem is powerful and gives away a simple way to evaluate integrals around arbitrary arcs.

Example . Find the integral of $f(z) = \frac{e^{iz}}{z}$ around $z = 0$ traversed contour-clockwise.

solution

Consider C_r to be a semi-circle of radius r centered at the origin then

$$\lim_{r\to 0}\int_{C_r}\frac{e^{iz}}{z-0}dz = (0-\pi)ie^{i0} = -\pi i$$

Note the negative sign because we are starting at an angle π and up to 0.

22 Real integrals using contour integration

22.1 Trigonometric functions

In this section we consider solving integrals of functions of the form $f(\cos\theta, \sin\theta)$ over the interval $(0, 2\pi)$. This can be done by realizing that for any point satisfying $|z| = 1$ then we have

$$z + z^{-1} = 2\cos\theta$$

$$z - z^{-1} = 2i\sin\theta$$

Hence one can verify that

$$\oint_{|z|=1} f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} = \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$$

This can be done by parameterizing $|z| = 1$ as $e^{i\theta}$ on the interval $\theta \in (0, 2\pi)$. Hence one can use the residue theorem to compute the complex integral.

22.1.1 Example

Evaluate the following integral

$$\int_0^{2\pi} \frac{\sin\theta}{\sin\theta + 2} d\theta$$

solution

Hence we can show

$$\int_0^{2\pi} \frac{\sin\theta}{\sin\theta+2}d\theta = \oint_{|z|=1} \frac{\frac{z-z^{-1}}{2i}}{2+\frac{z-z^{-1}}{2i}}\frac{dz}{iz} = -i\oint_{|z|=1}\frac{z^2-1}{z(z^2+4iz-1)}dz$$

The function has three poles $0, -2i \pm \sqrt{3}i$ with the only poles inside the contour $0, (-2+\sqrt{3})i$

$$\text{Res}(f, 0) = \lim_{z\to 0}\frac{z^2-1}{z^2+4iz-1} = 1$$

$$\text{Res}(f, (-2+\sqrt{3})i) = \lim_{z\to(-2+\sqrt{3})i}(z-(-2+\sqrt{3})i)\frac{z^2-1}{z(z-(-2+\sqrt{3})i)(z-(-2-\sqrt{3})i)} = \frac{-(-2+\sqrt{3})^2-1}{-2\sqrt{3}(-2+\sqrt{3})} = -\frac{2}{\sqrt{3}}$$

Finally we get the value by summing the residues

$$-i\oint_{|z|=1}\frac{z^2-1}{z(z^2+4iz-1)}dz = 2\pi\left(1-\frac{2}{\sqrt{3}}\right)$$

22.2 Integrating around an ellipse

22.2.1 Example

Prove for $a, b > 0$

$$\int_0^{2\pi} \frac{dt}{a^2 \cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}$$

solution

Let us integrate the following function

$$f(z) = \frac{1}{z}$$

Around the ellipse

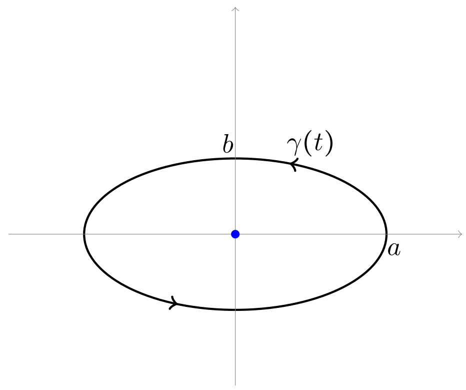


Figure 6: Integration around ellipse

By the residue theorem

$$\oint_\gamma f(z)\, dz = 2\pi i \operatorname{Res}(f, 0)$$

The parameterization of the ellipse $\gamma(t) = a\cos(t) + ib\sin(t)$

$$\oint_\gamma f(z)\, dz = \int_0^{2\pi} \frac{-a\sin t + ib\cos t}{a\cos t + ib\sin t}\, dt$$

The intgrand simplifies to the following

$$\frac{-a\sin t + ib\cos t}{a\cos t + ib\sin t} \times \frac{a\cos t - ib\sin t}{a\cos t - ib\sin t} = \frac{(b^2 - a^2)\sin t\cos t + iab}{a^2\cos^2 t + b^2\sin^2 t}$$

Hence

$$\int_0^{2\pi} \frac{(b^2 - a^2)\sin t\cos t + iab}{a^2\cos^2 t + b^2\sin^2 t}\, dt = 2\pi i \operatorname{Res}(f, 0) = 2\pi i$$

By equating the imaginary parts

$$iab \int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2 \sin^2 t} = 2\pi i$$

Which implies the result

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2 \sin^2 t} = \frac{2\pi}{ab}$$

22.3 Creating crazy integrals

Theorem. Let f be analytic function in the unit circle $|z| \leq 1$ such that $f \neq 0$. Then

$$\int_0^{2\pi} f(e^{it})\, dt = 2\pi\, f(0)$$

proof

Since the function f is analytic in and on the contour we have by the Cauchy integral theorem

$$\oint_{|z|=1} \frac{f(z)}{z}\, dz = 2\pi i \operatorname{Res}(f/z, 0)$$

Use $z = e^{it}$ whenere $0 \leq t \leq 2\pi$

$$i \int_0^{2\pi} \frac{f(e^{it})}{e^{it}} e^{it}\, dt = 2\pi i \operatorname{Res}(f/z, 0)$$

Note that

$$\operatorname{Res}\left(\frac{f(z)}{z}, 0\right) = \lim_{z\to 0} z \frac{f(z)}{z} = f(0)$$

Hence

$$\int_0^{2\pi} f(e^{it})\, dt = 2\pi\, f(0)$$

22.3.1 Example

Consider the function

$$f(z) = \exp(\exp(z))$$

It then follows from the theorem

$$\int_0^{2\pi} \exp(\exp(\exp(it)))\, dt = 2\pi\, e$$

Note that

$$\begin{aligned}
\exp(\exp(\exp(it))) &= e^{e^{\cos t + i \sin t}} \\
&= e^{e^{\cos t}(\cos(\sin t) + i \sin(\sin t))} \\
&= e^{e^{\cos t}\cos(\sin t)}\left(\cos(e^{\cos t}\sin(\sin t)) + i \sin(e^{\cos t}\sin(\sin t))\right)
\end{aligned}$$

By taking the real part

$$\int_0^{2\pi} e^{e^{\cos t}\cos(\sin t)} \cos(e^{\cos t}\sin(\sin t))\, dt = 2\pi\, e$$

22.3.2 Example

$$f(z) = \log(2+z)$$

Where we take the principal logarithm with the branch cut located at $y = 0, x \le -2$

$$\int_0^{2\pi} \log(2+e^{it}) = 2\pi \log(2)$$

Note that

$$\begin{aligned}
\log(2+e^{it}) &= \log|(2+\cos t)^2 + \sin^2 t| + i \arctan\left(\frac{\sin t}{1+\cos t}\right) \\
&= \frac{1}{2}\log(5+4\cos t) + i \arctan\left(\frac{\sin t}{2+\cos t}\right)
\end{aligned}$$

It follows by taking the real parts

$$\int_0^{2\pi} \log(5+4\cos t)\, dt = 4\pi \log(2)$$

22.3.3 Example

By combining the two functions

$$f(z) = \exp(\exp(z))\log(2+z)$$

We deduce that

$$\int_0^{2\pi} \exp(e^{\cos t}\cos(\sin t))(\cos(e^{\cos t}\sin(\sin t))\log(5+4\cos t) - 2\arctan\left(\frac{\sin t}{2+\cos t}\right)\sin(e^{\cos t}\sin(\sin t)))\,dt = 4\pi e\log(2)$$

22.4 Trigonometric functions with rationals of polynomials

Here we are interested in integrals of the form

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{P(x)}dx$$

where $P(x)$ is continuous in the real line and of degree 2 or more. You may notice that it is enough to look at the following function

$$f(z) = \frac{e^{iaz}}{P(z)}$$

and then take the real part. Generally speaking to solve that we take a semi-circle in the upper half plane where the contour will enclose some or all of the zeros of $P(z)$. Notice that we may use the Jordan's lemma to prove that the circular part goes to zero and what is left is the real integral along the real axis.

22.4.1 Example

Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^2+1}dx$$

proof

Let use consider the function with $a > 0$

$$f(z) = \frac{e^{iaz}}{z^2+1}$$

and integrate around the contour in Figure 7

We can write the integral using the residue theorem as

$$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}dx + \int_{C_R} f(z)dz = 2\pi i \operatorname{Res}(f, i)$$

By taking the limit $R \to \infty$ note that by the Jordan's lemma the integral along the circular part vanishes (prove it). Hence we have

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx = 2\pi i \operatorname{Res}(f, i)$$

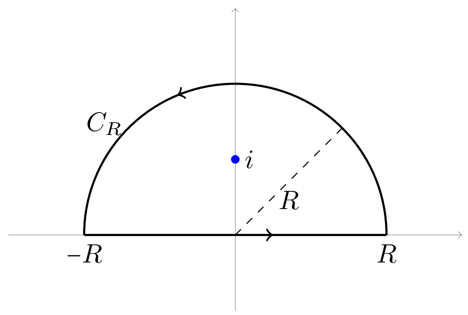


Figure 7: Circular contour

The evaluation of the residues

$$\operatorname{Res}(f,i) = \lim_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)} = \frac{e^{-a}}{2i}$$

Finally we get

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+1}dx = \pi e^{-a}$$

and eventually

$$\int_{-\infty}^{\infty}\frac{\cos(ax)}{x^2+1}dx = \pi e^{-a}$$

Advice. One might ask how to choose the complex function and the contour. There is no general formula for that and you can obtain that by experience and trail and error. It is actually a good idea to try different contours and see why they work!

22.5 Integration along contours with detours

Assume that we need to integrate a function and we have a pole at the path of integration then we have to make a small detour around the pole with some angle.

22.5.1 Example

Prove the following

$$\int_{0}^{\infty}\frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

proof

Consider the following function

$$f(z) = \frac{e^{iz}}{z}$$

Note that in order to integrate a round a semi-circle in the upper half plane we have to make a detour around the pole at $z = 0$

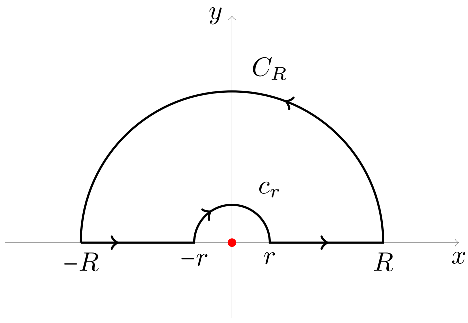


Figure 8: Integral with detour

Hence by the residue theorem

$$\int_r^R \frac{e^{ix}}{x}dx + \int_{-R}^{-r} \frac{e^{ix}}{x}dx + \int_{C_R} f(z)dz + \int_{C_r} f(z)dz = 0$$

Note first by taking $R \to \infty$ we have by the Jordan's lemma

$$\lim_{R\to\infty} \int_{C_r} f(z)dz = 0$$

Also we know how to deal with detours around poles

$$\lim_{r\to 0} \int_{C_r} \frac{e^{iz}}{z-0}dz = -\pi i e^{0i} = -\pi i$$

Hence we finally get

$$\int_0^\infty \frac{e^{ix}}{x}dx + \int_{-\infty}^0 \frac{e^{ix}}{x}dx = \pi i$$

By inverting the sign in the second integral

$$\int_0^\infty \frac{e^{ix} - e^{-ix}}{x}dx = 2i \int_0^\infty \frac{\sin(x)}{x}dx = \pi i$$

By rearranging we have our result.

22.6 Integrals of functions with branch cuts

We have a serious problem when dealing with functions like $f(z) = \log(z)$ or z^a where we have a branch point at 0. In order to evaluate the integral we have to avoid both the branch point and the branch cut. Generally speaking the integral around a branch point will most probably vanish but this is not taken for granted. You have actually to prove it first.

22.6.1 Example

Prove the following

$$\int_0^\infty \frac{\log(x)}{x^2+2} = \frac{\pi \log(2)}{4\sqrt{2}}$$

proof

Let us consider the usual integral of a semi-circle with a detour around the branch point at 0. But first we have to choose an analytic branch of the logarithm.

Consider the following branch

$$\log(z) = \log|z| + i\arg(z)$$

where $\arg(z) \in (-\pi/2, 3\pi/2)$ i.e we are taking the branch cut on the negative imaginary axis. By this construction we have an analytic function along the path of integration. Now consider the function

$$f(z) = \frac{\log(z)}{z^2+2}$$

The zero $z = \sqrt{2}i$ lies inside the semi-circle

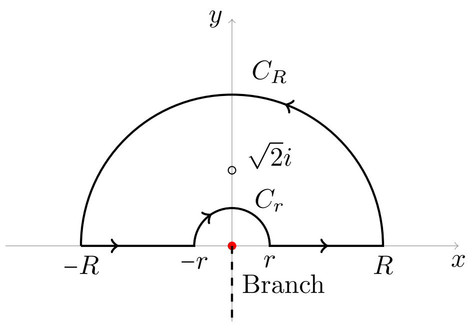


Figure 9: Integral of function with branch cut

By the residue theorem

$$\int_r^R \frac{\log|x|}{x^2+2}dx + \int_{-R}^{-r} \frac{\log|x|+i\pi}{x^2+2}dx + \int_{C_R} f(z)dz + \int_{C_r} f(z)dz = 2\pi i \text{Res}(f, \sqrt{2}i)$$

Note the $|\log(z)| = |\log|z| + i\arg(z)| \leq |\log R| + |\arg(z)|$. Since the argument is bounded we can bound the function using

$$\left|\frac{\log(z)}{z^2+2}\right| \leq \frac{|\log|z|| + Const}{|z|^2-2}$$

Which vanishes on the larger curve since

$$\lim_{R\to\infty} R\frac{\log R + Const}{R^2-2} = 0$$

Similarly we have

$$\lim_{r\to 0} r\frac{|\log r| + Const}{r^2-2} = 0$$

Hence it follows by the Estimation lemma that both integrals vanish and

$$\int_0^\infty \frac{\log|x|}{x^2+2}dx + \int_{-\infty}^0 \frac{\log|x|+i\pi}{x^2+2}dx = 2\pi i \text{Res}(f, \sqrt{2}i)$$

By rearranging

$$2\int_0^\infty \frac{\log|x|}{x^2+2}dx + \pi i\int_0^\infty \frac{dx}{x^2+2} = 2\pi i \text{Res}(f, \sqrt{2}i)$$

The residue evaluation

$$\text{Res}(f, \sqrt{2}i) = \frac{\log(\sqrt{2}i)}{2\sqrt{2}i} = \frac{\log\sqrt{2} + \frac{\pi}{2}i}{2\sqrt{2}i}$$

Hence

$$2\int_0^\infty \frac{\log|x|}{x^2+2}dx + \pi i\int_0^\infty \frac{dx}{x^2+2} = \pi\frac{\log\sqrt{2} + \frac{\pi}{2}i}{\sqrt{2}}$$

By equating the real parts we get our result.

22.6.2 Example

Prove the following

$$\int_0^\infty \frac{\log\left(x^2+1\right)\arctan^2(x)}{x^2}\,dx = \frac{\pi^3}{12} + \pi\log^2(2)$$

solution

Lemma

$$\int_0^\infty \frac{\log^3(1+x^2)}{x^2}\,dx = \pi^3 + 3\pi\log^2(4)$$

proof

Start by the following

$$\int_0^\infty x^{-p}(1+x)^{s-1}dx = \frac{\Gamma(1-p)\Gamma(p-s)}{\Gamma(1-s)}$$

Let $x \to x^2$

$$\int_0^\infty x^{-2p+1}(1+x^2)^{s-1}dx = \frac{\Gamma(1-p)\Gamma(p-s)}{2\Gamma(1-s)}$$

Let $p = 3/2$

$$\int_0^\infty \frac{1}{x^2(1+x^2)^{1-s}}dx = \frac{\Gamma(-1/2)\Gamma(3/2-s)}{2\Gamma(1-s)}$$

By taking the third derivative and $s \to 1$

$$\int_0^\infty \frac{\log^3(1+x^2)}{x^2}\,dx = \frac{\Gamma(-1/2)}{2}\left[\frac{d^3}{ds^3}\frac{\Gamma(3/2-s)}{2\Gamma(1-s)}\right]_{s=1} = \pi^3 + 3\pi\log^2(4)$$

Now, Consider the function

$$f(z) = \frac{\log^3(1-iz)}{z^2}$$

Define the principle logarithm as follows

$$\log z = \log|z| + i\mathrm{Arg}(z)$$

Note that for $x > 0$ the argument can be evaluated as

$$\log z = \log\sqrt{x^2+y^2} + i\arctan(y/x)$$

For the principle logarithm the branch cut is defined as $\Im(1-iz) = 0, \Re(1-iz) \le 0$ which then reduces for $z = x+iy$ to $x = 0, 1+y \le 0$. Hence the branch cut on the imaginary axis where $x = 0, y < -1$. Also note that $f(z)$ is analytic on the punctured plane since around $z = 0$

$$\frac{\log^3(1-iz)}{z^2} = -\frac{((iz)+(iz)^2/2+(iz)^3/3+\mathcal{O}(z^4))^3}{z^2} = \mathcal{O}(z^2)$$

Hence at $z = 0$ we have a removable singularity. Define the following contour were we avoid the branch cut

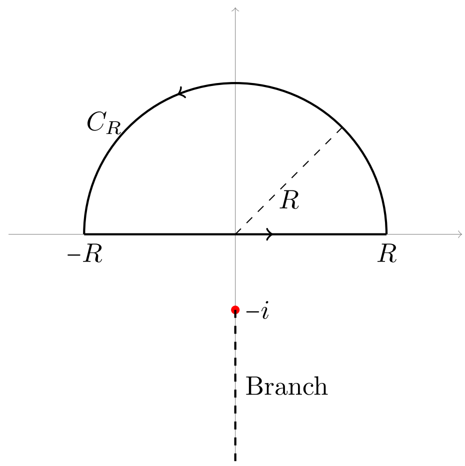


Figure 10: Integral with branch cut

$$\int_{C_R} f(z)\,dz + \int_{-R}^{0} \frac{\left(\log(\sqrt{1+x^2}) + i\arctan(x)\right)^3}{x^2}dx + \int_0^R \frac{(\log(\sqrt{1+x^2}) + i\arctan(x))^3}{x^2}dx = 0$$

Apparently

$$\int_{C_R} f(z)\,dz + \int_{0}^{R} \frac{\left(\log(\sqrt{1+x^2}) - i\arctan(x)\right)^3}{x^2}dx + \int_0^R \frac{(\log(\sqrt{1+x^2}) + i\arctan(x))^3}{x^2}dx = 0$$

Note that

$$(x+y)^3 = x^3 + y^3 + 3x^2y + 3yx^2$$

This simplifies to

$$\int_{C_R} f(z)\,dz + \int_0^R \frac{-3\arctan^2(x)\log(1+x^2) + 1/4\log^3(1+x^2)}{x^2}\,dx = 0$$

Taking the limit

$$\left|\int_{C_R} f(z)\,dz\right| = \left|\int_{C_R} \frac{\log^3(1-iz)}{z^2}\,dz\right| \leq \pi R\frac{(\log|1+R| + 2\pi)^3}{R^2} \sim_\infty 0$$

So this simplifies to

$$\int_0^\infty \frac{\arctan^2(x)\log(1+x^2)}{x^2}\,dx = \frac{1}{12}\int_0^\infty \frac{\log^3(1+x^2)}{x^2}\,dx$$

Using the Lemma we reach our result.

22.6.3 Example

Prove $a, b, c, d > 0$

$$\int_0^\infty \frac{\log(a^2+b^2x^2)}{c^2+d^2x^2}\,dx = \frac{\pi}{cd}\log\frac{ad+bc}{d}$$

proof

Consider the function

$$f(z) = \frac{\log(a-ibz)}{c^2+d^2z^2}$$

We need the logarithm with the branch cut $y < -\frac{a}{b}, x = 0$. Note that this corresponds to

$$\log(a+ibz) = \log\sqrt{(a+y)^2+b^2x^2} + i\theta \quad , \theta \in [-\frac{\pi}{2}, \frac{3\pi}{2})$$

Consider the contour that avoids the branch-cut on the negative imaginary part

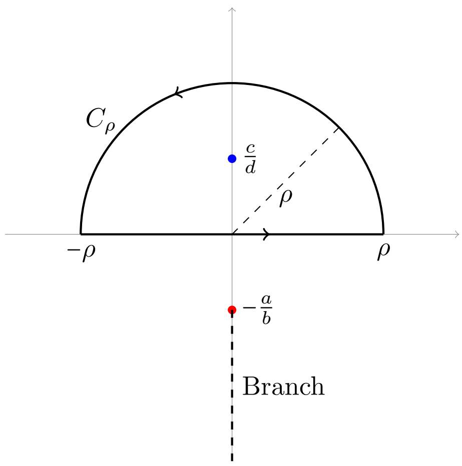


Figure 11: Contour with branch cut on the imaginary axis

$$\int_{C_\rho} f(z)\,dz + \int_0^\rho \frac{\log(a-ibx)}{c^2+dx^2}\,dx + \int_{-\rho}^0 \frac{\log(a-ibx)}{c^2+dx^2}\,dx = 2\pi i \,\mathrm{Res}\left(f, \frac{c}{d}i\right)$$

In the second integral let $x \to -x$

$$\int_{C_\rho} f(z)\,dz + \int_0^\rho \frac{\log(a-ibx)}{c^2+dx^2}\,dx + \int_0^\rho \frac{\log(a+ibx)}{c^2+dx^2}\,dx = 2\pi i \,\mathrm{Res}\left(f, \frac{c}{d}i\right)$$

Note that in the x-axis we have

$$\log(a \pm bix) = \log(\sqrt{a^2+b^2x^2}) \pm i\arctan\left(\frac{bx}{a}\right)$$

Using that we deduce

$$\log(a - bix) + \log(a + ibx) = \log(a^2 + b^2x^2)$$

This implies to

$$\int_{C_\rho} f(z)\,dz + \int_0^\rho \frac{\log(a^2 + b^2x^2)}{c^2 + dx^2}\,dx = 2\pi i\,\text{Res}\left(f, \frac{c}{d}i\right)$$

For the circular part

$$\left|\int_{C_\rho} \frac{\log(a - ibz)}{c^2 + d^2z^2}\,dz\right| \le \pi\rho\,\frac{\log(a + b\rho) + 2\pi}{|c^2 - d^2\rho^2|} \sim_\infty 0$$

Let us look at the residue

$$2\pi i\text{Res}\left(f, \frac{c}{d}i\right) = 2\pi i \lim_{z\to\frac{c}{d}i} \frac{\log(a - ibz)}{2d^2z} = \frac{\pi}{cd}\log\frac{ad + bc}{d}$$

Hence this simplifies to

$$\int_0^\infty \frac{\log(a^2 + b^2x^2)}{c^2 + dx^2}\,dx = \frac{\pi}{cd}\log\frac{ad + bc}{d}$$

22.6.4 Example

Prove the following

$$\int_0^\infty \frac{\log(x)\cos(x)}{(x^2 + 1)^2}\,dx = -\frac{\pi\text{Ei}(1)}{4e} - \frac{\pi}{4e}$$

proof

Consider the following function

$$f(z) = \frac{\log(z)}{(z^2 + 1)^2}e^{iz}$$

Now consider the the branch of the logarithm

$$\log(z) = \log|r| + i\theta \ \ ,\theta \in (-\pi/2, 3\pi/2]$$

Consider the contour in Figure 12

Then by the residue theorem

$$\int_{C_R} f(z)\,dz + \int_{c_r} f(z)\,dz + \int_r^R \frac{\log(x)}{(x^2 + 1)^2}e^{ix}\,dx + \int_{-R}^{-r} \frac{(\log|x| + \pi i)}{(x^2 + 1)^2}e^{ix}\,dx = 2\pi i\,\text{Res}(f, i)$$

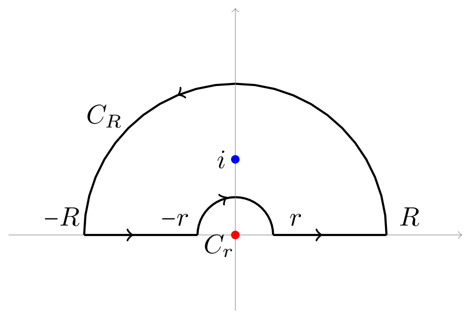


Figure 12: Contour with detour around 0

Which simplifies to

$$\int_{C_R} f(z)\,dz + \int_{c_r} f(z)\,dz + \int_r^R \frac{\log(x)}{(x^2+1)^2} e^{ix}\,dx + \int_r^R \frac{(\log(x)+\pi i)}{(x^2+1)^2} e^{-ix}\,dx = 2\pi i \text{Res}(f,i)$$

Or

$$\int_{C_R} f(z)\,dz + \int_{c_r} f(z)\,dz + 2\int_r^R \frac{\cos(x)\log(x)}{(x^2+1)^2}\,dx + \pi i \int_r^R \frac{e^{-ix}}{(x^2+1)^2}\,dx = 2\pi i \text{Res}(f,i)$$

Note that for the semi-circle $Re^{i\theta}$

$$M_R = \max_{\theta\in[0,\pi]} \left| \frac{\log(R)+i\theta}{(R^2e^{2i\theta}+1)^2} \right| \leq \frac{\log R + 2\pi}{(R^2-1)^2}$$

It follows by the Jordan's lemma

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

For the smaller semi-circle

$$\int_{c_r} f(z)\,dz = ir\int_0^\pi \frac{\log r + i\pi}{(r^2e^{2i\theta}+1)^2} e^{re^{i\theta}} e^{i\theta}\,d\theta$$

Note that by a similar argument to the Jordan's lemma we have

$$\lim_{r\to 0} \left| \int_{c_r} f(z)\,dz \right| \leq \lim_{r\to 0} \pi r M_r \to 0$$

It follows then

$$2\int_0^\infty \frac{\cos(x)\log(x)}{(x^2+1)^2}\,dx + \pi i \int_0^\infty \frac{e^{-ix}}{(x^2+1)^2}\,dx = 2\pi i \text{Res}_i f(z)$$

Evaluating the residue

$$\text{Res}_i f(z) = \lim_{z \to i} \frac{d}{dz}\left((z-i)^2 \frac{\log(z)}{(z^2+1)^2} e^{iz}\right) = \frac{\pi + i}{4e}$$

Note that

$$\int_0^\infty \frac{e^{ix}}{(x^2+1)^2}\, dx = \frac{\pi}{2e} - i\frac{\text{Ei}(1)}{2e}$$

Since

$$2\int_0^\infty \frac{\cos(x)}{(x^2+1)^2}\, dx = 2\pi i \left(\frac{-i}{2e}\right)$$

By integrating around a semi-circle in the upper half plane

$$\int_0^\infty \frac{\cos(x)}{(x^2+1)^2}\, dx = \frac{\pi}{2e}$$

For the other integral, It is easy to see that

$$\int_0^\infty \frac{\sin(x)}{x^2+a^2}\, dx = \frac{1}{2a}[e^{-a}\text{Ei}'(a) - e^{a}\text{Ei}(-a)]$$

By differentiation and letting $a \to 1$ we have

$$\int_0^\infty \frac{\sin(x)}{(x^2+1)^2}\, dx = \frac{\text{Ei}(1)}{2e}$$

We deduce that

$$2\int_0^\infty \frac{\log(x)\cos(x)}{(x^2+1)^2}\, dx + \pi i\left(\frac{\pi}{2e} - i\frac{\text{Ei}(1)}{2e}\right) = 2\pi i\left(\frac{\pi}{4e} + \frac{i}{4e}\right)$$

This simplifies to

$$\int_0^\infty \frac{\log(x)\cos(x)}{(x^2+1)^2}\, dx = -\frac{\pi \text{Ei}(1)}{4e} - \frac{\pi}{4e}$$

22.6.5 Example

Prove the following

$$\int_0^\infty \frac{x^\alpha}{x+1}\, dx = -\pi \csc(\pi\alpha)$$

proof

Consider the following function

$$f(z) = \frac{z^\alpha}{1+z} = \frac{e^{\alpha \log(z)}}{1+z}$$

We consider the branch of the logarithm

$$\log(z) = \ln|z| + i\theta\,,\ \theta \in [0, 2\pi)$$

This function is single-valued and on the integration path we have

$$e^{\alpha \log(z)} = x^\alpha e^{i\theta\alpha} = \begin{cases} x^\alpha & \theta \to 0 \\ x^\alpha e^{2\pi\alpha i} & \theta \to 2\pi \end{cases}$$

Integrating around contour (called key-hole contour) as in Figure 13

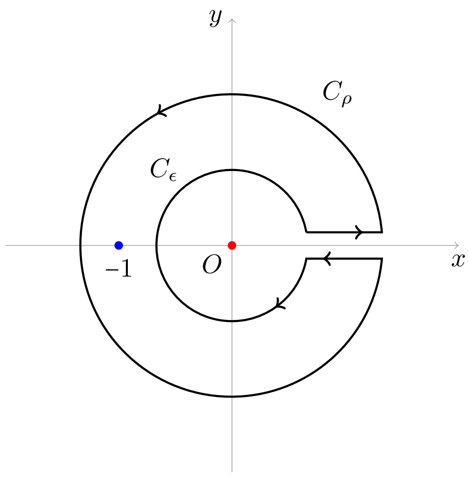


Figure 13: Key hole contour

By the residue theorem we have

$$\oint f(z)\,dz = 2\pi i \operatorname{Res}(f, -1)$$

Which becomes

$$\int_{C_\rho} f(z)\,dz + \int_{C_\epsilon} f(z)\,dz + \int_\epsilon^\rho f(z)dz - \int_\epsilon^\rho f(z)dz = 2\pi i \operatorname{Res}(f, -1)$$

By the chosen branch of the algorithm we have

$$\int_{C_\rho} f(z)\,dz + \int_{C_\epsilon} f(z)\,dz + \int_\epsilon^\rho \frac{x^\alpha}{x+1}\,dx - e^{2\pi i\alpha} \int_\epsilon^\rho \frac{x^\alpha}{x+1}\,dx = 2\pi i \operatorname{Res}(f, -1)$$

Note for the circle $|z| = \rho$ we have

$$\left|\oint_{|z|=\rho} f(z)\,dz\right| \le 2\pi\rho \max\left|\frac{z^\alpha}{1+z}\right| \le 2\pi\rho\frac{|z|^\alpha}{||z|-1|} = 2\pi\frac{\rho^{\alpha+1}}{|\rho-1|}$$

Note the triangle inequality

$$|\omega+1| \ge ||\omega|-|1||$$

Note that if $-1 < \alpha < 0$ we get

$$\left|\lim_{\rho\to\infty}\oint_{|z|=\rho} f(z)\,dz\right| \le 2\pi\frac{\rho^{\alpha+1}}{|\rho-1|} \to 0$$

Similarly we have

$$\left|\lim_{\epsilon\to 0}\oint_{|z|=\epsilon} f(z)\,dz\right| \le 2\pi\frac{\epsilon^{\alpha+1}}{|\epsilon-1|} \to 0$$

We deduce that as $\epsilon \to 0$ and $\rho \to \infty$

$$(1-e^{2\pi i\alpha})\int_0^\infty \frac{x^\alpha}{x+1}\,dx = 2\pi i\,\mathrm{Res}(f,-1) \ \ , -1 < \alpha < 0$$

Notice that

$$\mathrm{Res}(f,-1) = \lim_{z\to -1} e^{\alpha\log(z)} = e^{\alpha(\ln|-1|+\pi i)} = e^{\alpha\pi i}$$

Hence we have

$$\int_0^\infty \frac{x^\alpha}{x+1}\,dx = 2\pi i\,\frac{e^{\alpha\pi i}}{1-e^{2\alpha\pi i}} = \frac{2\pi i}{e^{-\pi i\alpha}-e^{\pi i\alpha}} = -\pi\csc(\pi\alpha)$$

Note we can deduce the Euler reflection formula

$$\Gamma(\alpha)\Gamma(1-\alpha) = \pi\csc(\pi\alpha)$$

22.6.6 Example

Prove the following

$$\int_0^{\pi/2} \cos(nt)\cos^m(t)\,dt = \frac{\pi\Gamma(m+1)}{2^{m+1}\Gamma\left(\frac{n+m+2}{2}\right)\Gamma\left(\frac{2-n+m}{2}\right)}$$

proof

Let us integrate the following function

$$f(z) = z^{n-m-1}\left(1+z^2\right)^m$$

We choose the principle logarithm where

$$\log(z) = \log|z| + \operatorname{Arg}(z)$$

Note that the function $z^{n-m-1} = e^{(n-m-1)\log(z)}$ will have a branch cut on the negative x axis. Also we have

$$\left(1+z^2\right)^m = e^{m\log(1+z^2)}$$

The principle branch will be on the imaginary y axis where $|y| \geq 1$ which implies that $y \geq 1$ or $y \leq -1$. We integrate around the semi-circle of radius 1 as in Figure 14

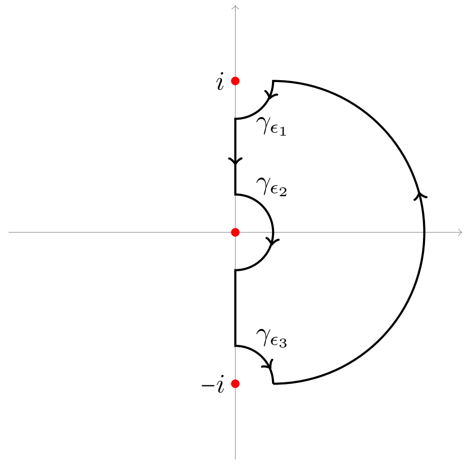


Figure 14: Semi-circle indented at the branch points

By the residue theorem

$$\int_{\gamma_{\epsilon_1}} f(z)\,dz + \int_{\gamma_{\epsilon_2}} f(z)\,dz + \int_{\gamma_{\epsilon_3}} f(z)\,dz + \int_{i-i\epsilon_1}^{i\epsilon_2} f(z)\,dz + \int_{-i\epsilon_2}^{-i+i\epsilon_3} f(z)\,dz + \int_C f(z)\,dz = 0$$

I'll prove the second integral goes to 0

Use the parameterization $\gamma_{\epsilon_2}(t) = \epsilon_2 e^{it}$, $-\pi/2 \leq t \leq \pi/2$

$$\left| i\epsilon_2^{n-m} \int_{-\pi/2}^{\pi/2} e^{int} \log(1+(\epsilon_2 e^{it})^2)\,dt \right| \leq \pi \log(2)\epsilon_2^{n-m}$$

By taking the limit we deduce the integral goes to 0. Similarly we have

$$\lim_{\epsilon_1 \to 0} \int_{\gamma_{\epsilon_1}} f(z)\,dz = \lim_{\epsilon_2 \to 0} \int_{\gamma_{\epsilon_3}} f(z)\,dz = 0$$

Note on on the circular part of $|z| = 1$

$$\begin{aligned}\int_C f(z)\,dz &= i\int_{-\pi/2}^{\pi/2} e^{it}e^{int-imt-it}\left(1+e^{2it}\right)^m\,dt\\ &= i\int_{-\pi/2}^{\pi/2} e^{int}(e^{-it}+e^{it})^m\,dt\\ &= i2^m\int_{-\pi/2}^{\pi/2} e^{int}\cos^m(t)\,dt\end{aligned}$$

The integrals reduce to

$$\int_i^0 x^{n-m-1}\left(1+x^2\right)^m\,dx + \int_0^{-i} x^{n-m-1}\left(1+x^2\right)^m\,dx = -i2^m\int_{-\pi/2}^{\pi/2} e^{int}\cos^m(t)\,dt$$

By taking $x = it$ and $x = -it$ respectively

$$-i\int_0^1 (it)^{n-m-1}\left(1-t^2\right)^m\,dt - i\int_0^1 (-it)^{n-m-1}\left(1-t^2\right)^m\,dt = -i2^m\int_{-\pi/2}^{\pi/2} e^{int}\cos^m(t)\,dt$$

We then can combine the integrals

$$-i((i)^{n-m-1}+(-i)^{n-m-1})\int_0^1 t^{n-m-1}\left(1-t^2\right)^m\,dt = -i2^m\int_{-\pi/2}^{\pi/2} e^{int}\cos^m(t)\,dt$$

Note that

$$-(i^{n-m}-(-i)^{n-m}) = -\left(e^{i(n-m)\pi/2}-e^{-i(n-m)\pi/2}\right) = -2i\sin\left(\frac{n\pi-m\pi}{2}\right)$$

We deduce that

$$\int_{-\pi/2}^{\pi/2} e^{int}\cos^m(t)\,dt = 2^{1-m}\sin\left(\frac{n\pi-m\pi}{2}\right)\int_0^1 t^{n-m-1}(1-t^2)^m\,dt$$

Using the Euler formula we have

$$\int_0^{\pi/2}\cos(nt)\cos^m(t)\,dt = 2^{-m}\sin\left(\frac{n\pi-m\pi}{2}\right)\int_0^1 t^{n-m-1}(1-t^2)^m\,dt$$

Note the beta integral

$$\int_0^1 t^{n-m-1}(1-t^2)^m\,dt = \frac{\Gamma(m+1)\Gamma\left(\frac{n-m}{2}\right)}{2\Gamma\left(\frac{n+m+2}{2}\right)}$$

By the reflection formula we have

$$\Gamma\left(\frac{n-m}{2}\right)\Gamma\left(1-\frac{n-m}{2}\right) = \frac{\pi}{\sin\left(\frac{n\pi-m\pi}{2}\right)}$$

We deduce then that

$$\int_0^{\pi/2} \cos(nt)\cos^m(t)\,dt = \frac{\pi\Gamma(m+1)}{2^{m+1}\Gamma\left(\frac{n+m+2}{2}\right)\Gamma\left(\frac{2-n+m}{2}\right)}$$

22.7 Rectangular contours

In this section we consider the case of a rectangular contour. Usually considering such contours we have to separate the contour into four integrals. Note that such contours are useful for evaluating integrals of hyperbolic functions and infinite series in general.

22.7.1 Example

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{\cosh(x)}\,dx = \pi\,\mathrm{sech}\left(\frac{\pi a}{2}\right)$$

proof

Consider

$$f(z) = \frac{e^{iaz}}{\sinh(z)}$$

If we integrate around a contour of height π and stretch it to infinity we get

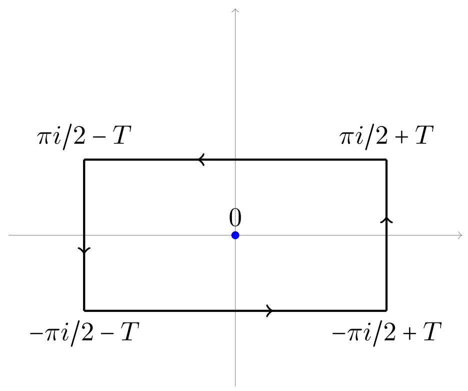


Figure 15: Rectangular contour

By taking $T \to \infty$

$$\int_{-i\pi/2+\infty}^{i\pi/2+\infty} f(x)\,dx + \int_{i\pi/2+\infty}^{i\pi/2-\infty} f(x)\,dx + \int_{i\pi/2-\infty}^{-i\pi/2-\infty} f(x)\,dx + \int_{-i\pi/2-\infty}^{-i\pi/2+\infty} f(x)\,dx = 2\pi i \mathrm{Res}(f,0)$$

Consider

$$\int_{-i\pi/2-\infty}^{-i\pi/2+\infty} \frac{e^{iax}}{\sinh(x)}\,dx$$

Let $x = -\pi i/2 + y$

$$ie^{\frac{\pi a}{2}} \int_{-\infty}^{\infty} \frac{e^{iay}}{\cosh(y)}\, dy$$

Similarly we have for

$$\int_{i\pi/2+\infty}^{i\pi/2-\infty} \frac{e^{iax}}{\sinh(x)}\, dx$$

By letting $x = \pi i/2 + y$

$$ie^{-\frac{\pi a}{2}} \int_{-\infty}^{\infty} \frac{e^{iay}}{\cosh(y)}\, dy$$

Now consider the sum of the two remaining integrals

$$\int_{-i\pi/2+\infty}^{i\pi/2+\infty} \frac{e^{iax}}{\sinh(x)} dx + \int_{i\pi/2-\infty}^{-i\pi/2-\infty} \frac{e^{iax}}{\sinh(x)}\, dx$$

Let $x \to -x$ in the second integral and summing

$$\int_{-i\pi/2+\infty}^{i\pi/2+\infty} \frac{e^{iax} - e^{-iax}}{\sinh(x)} dx = 2i \int_{-i\pi/2+\infty}^{i\pi/2+\infty} \frac{\sin(ax)}{\sinh(x)} dx$$

Let consider for simplicity

$$\int_{-i\pi/2+T}^{i\pi/2+T} \frac{\sin(x)}{\sinh(x)} dx$$

Let $y = -i(x - T)$

$$-i \int_{-i\pi/2}^{\pi/2} \frac{\sin(i(y+T))}{\sinh(i(y+T))} dy = -i \int_{-\pi/2}^{\pi/2} \frac{\sinh(y+T)}{\sin(y+T)} dy = -i \int_{-\pi/2+T}^{\pi/2+T} \frac{\sinh(y)}{\sin(y)} dy$$

Hence by taking $T \to \infty$ we are integrating around an infinitely small area which goes to 0

$$i\left(e^{\frac{\pi a}{2}} + e^{-\frac{\pi a}{2}}\right) \int_{-\infty}^{\infty} \frac{e^{iay}}{\cosh(y)}\, dy = 2\pi i \operatorname{Res}(f, 0)$$

Calculating the residue we have

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} z \frac{e^{iaz}}{\sinh(z)} = \lim_{z\to 0} \frac{e^{iaz}}{\cosh(z)} = 1$$

Using that we get

$$\int_{-\infty}^{\infty} \frac{e^{iay}}{\cosh(y)}\, dy = \frac{2\pi}{e^{\frac{\pi a}{2}} + e^{-\frac{\pi a}{2}}}$$

By taking the real part

$$\int_{-\infty}^{\infty} \frac{\cos(ay)}{\cosh(y)}\, dy = \pi \operatorname{sech}\left(\frac{\pi}{2} a\right)$$

22.7.2 Example

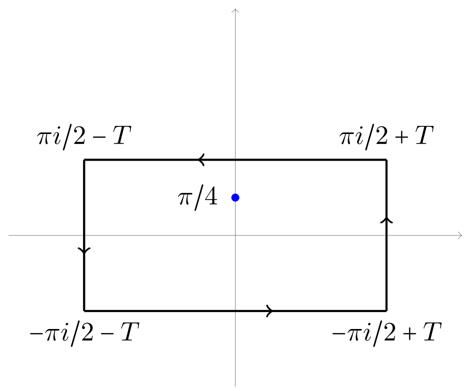
$$\int_{-\infty}^{\infty} \frac{1}{(5\pi^2 + 8\pi y + 16y^2)} \frac{\cosh\left(y + \frac{\pi}{4}\right)}{\cosh^3(y)} dy = \frac{2}{\pi^3}\left(\pi \cosh\left(\frac{\pi}{4}\right) - 4 \sinh\left(\frac{\pi}{4}\right)\right)$$

proof

Consider

$$f(z) = \frac{\sinh(z)}{z \sinh^3(z - \pi/4)}$$

If we integrate around a contour of height π and stretch it to infinity we get



By taking $T \to \infty$

$$\int_{-i\pi/2+\infty}^{i\pi/2+\infty} f(x)\, dx + \int_{i\pi/2+\infty}^{i\pi/2-\infty} f(x)\, dx + \int_{i\pi/2-\infty}^{-i\pi/2-\infty} f(x)\, dx + \int_{-i\pi/2-\infty}^{-i\pi/2+\infty} f(x)\, dx = 2\pi i \mathrm{Res}(f, \frac{\pi}{4})$$

Consider

$$\int_{-i\pi/2-\infty}^{-i\pi/2+\infty} \frac{\sinh(x)}{x \sinh^3(x - \pi/4)}\, dx$$

Let $x = -\pi/2i + \pi/4 + y$

$$-\int_{-\infty}^{\infty} \frac{1}{-i\pi/2 + \pi/4 + y} \frac{\cosh(\pi/4 + y)}{\cosh^3(y)}\, dy$$

Similarly we have for

$$\int_{i\pi/2+\infty}^{i\pi/2-\infty} \frac{\sinh(x)}{x \sinh^3(x - \pi/4)}\, dx$$

By letting $x = i\pi/2 + \pi/4 + y$

$$\int_{-\infty}^{\infty} \frac{1}{i\pi/2 + \pi/4 + y} \frac{\cosh(\pi/4 + y)}{\cosh^3(y)}\, dy$$

The other integrals go to 0 hence

$$-16\pi i \int_{-\infty}^{\infty} \frac{1}{(5\pi^2 + 8\pi y + 16y^2)} \frac{\cosh\left(y + \frac{\pi}{4}\right)}{\cosh^3(y)} dy = 2\pi i \mathrm{Res}(f, \frac{\pi}{4})$$

Calculating the residue we have

$$-16\pi i \int_{-\infty}^{\infty} \frac{1}{(5\pi^2 + 8\pi y + 16y^2)} \frac{\cosh\left(y + \frac{\pi}{4}\right)}{\cosh^3(y)} dy = 2\pi i \frac{-(16(\pi\cosh(\pi/4) - 4\sinh(\pi/4))}{\pi^3}$$

Which reduces to our result

$$\int_{-\infty}^{\infty} \frac{1}{(5\pi^2 + 8\pi y + 16y^2)} \frac{\cosh\left(y + \frac{\pi}{4}\right)}{\cosh^3(y)} dy = \frac{2}{\pi^3}\left(\pi\cosh\left(\frac{\pi}{4}\right) - 4\sinh\left(\frac{\pi}{4}\right)\right)$$

22.7.3 Example

Prove the following

$$\int_0^{\infty} \frac{\sin(ax)}{e^{2\pi x} - 1}\, dx = \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a}$$

solution

By integrating the following function

$$f(z) = \frac{e^{iaz}}{e^{2\pi z} - 1}$$

The function is analytic in and on the contour, indented at the poles of the function

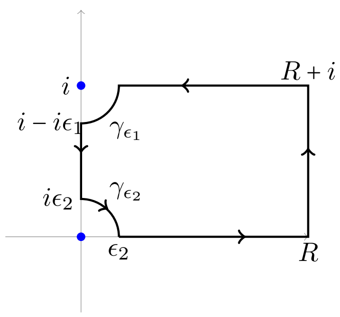


Figure 16: Indented rectangular contour

Hence by the residue theorem

$$\int_{\gamma_{\epsilon_2}} f(z)\,dz + \int_{\epsilon_2}^{R} f(x)\,dx + \int_{R}^{R+i} f(x)\,dx + \int_{R+i}^{i+\epsilon_2} f(x)\,dx + \int_{\gamma_{\epsilon_1}} f(z)\,dz - \int_{i\epsilon_2}^{i-i\epsilon_1} f(x)\,dx = 0$$

Let us first look at $R \to \infty$

The vertical line

$$\begin{aligned}
\left|\int_{R}^{R+i} \frac{e^{iax}}{e^{2\pi x}-1}\,dx\right| &= \left|iR\int_0^1 \frac{e^{ia(R+ixR)}}{e^{2\pi(R+ixR)}-1}\,dx\right| \\
&\leq R\int_0^1 \frac{|e^{ia(R+ixR)}|}{|e^{2\pi(R+ixR)}-1|}\,dx \\
&\leq \frac{R}{|e^{2\pi R}-1|}\int_0^1 e^{-axR}\,dx \\
&= \frac{1}{a|e^{2\pi R}-1|}(1-e^{-aR}) \sim_{\infty} 0
\end{aligned}$$

The other vertical line

$$\begin{aligned}
\int_{i\epsilon_2}^{i(1-\epsilon_1)} \frac{e^{iax}}{e^{2\pi x}-1}\,dx &= i\int_{\epsilon_2}^{(1-\epsilon_1)} \frac{e^{-ax}}{e^{2\pi ix}-1}\,dx \\
&= \frac{1}{2}\int_{\epsilon_2}^{(1-\epsilon_1)} \frac{e^{-ax}}{\sin(\pi x)e^{i\pi x}}\,dx \\
&= \frac{1}{2}\int_{\epsilon_2}^{(1-\epsilon_1)} \frac{e^{-ax}}{\sin(\pi x)e^{i\pi x}}\,dx \\
&= \frac{1}{2}\int_{\epsilon_2}^{(1-\epsilon_1)} \frac{\cos(\pi x)}{\sin(\pi x)}e^{-ax}\,dx - \frac{i}{2}\int_{\epsilon_2}^{(1-\epsilon_1)} e^{-ax}\,dx
\end{aligned}$$

Since the first integral diverges when $\epsilon_1, \epsilon_2 \to 0$

$$PV\int_0^i \frac{e^{iax}}{e^{2\pi x}-1}\,dx = PV\int_0^1 \frac{\cos(\pi x)}{2\sin(\pi x)}e^{-ax}\,dx - i\frac{1-e^{-a}}{2a}$$

The remaining integrals can be evaluated using residues

$$\lim_{\epsilon_2\to 0}\int_{\gamma_{\epsilon_2}} f(z)\,dz = -\frac{\pi i}{4}\text{Res}(f,0) = -\frac{i}{2}$$

$$\lim_{\epsilon_1\to 0}\int_{\gamma_{\epsilon_1}} f(z)\,dz = -\frac{\pi\, i}{4}\text{Res}(f,i) = -\frac{i}{2}e^{-a}$$

By combining the results together

$$PV\int_0^{\infty} \frac{e^{iax}}{e^{2\pi x}-1}\,dx - PV\int_0^{\infty} \frac{e^{ia(x+i)}}{e^{2\pi(x+i)}-1}\,dx - PV\int_0^1 \frac{\cos(\pi x)}{2\sin(\pi x)}e^{-ax}\,dx + i\frac{1-e^{-a}}{2a} = i\frac{e^{-a}+1}{4}$$

Which reduces to

$$(1-e^{-a})PV\int_0^\infty \frac{e^{iax}}{e^{2\pi x}-1}\,dx - PV\int_0^1 \frac{\cos(\pi x)}{2\sin(\pi x)}e^{-ax}\,dx = i\frac{e^{-a}+1}{4} - i\frac{1-e^{-a}}{2a}$$

By equating the imaginary parts

$$\int_0^\infty \frac{\sin(ax)}{e^{2\pi x}-1}\,dx = \frac{1}{4}\coth\left(\frac{a}{2}\right) - \frac{1}{2a}$$

22.8 Triangular contours

22.8.1 Example

$$\int_0^\infty \frac{\cos(x)}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin(x)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

solution

Choose the function with the principal logarithm

$$f(z) = z^{-1/2}\,e^{iz} = e^{-1/2\log(z)+iz}$$

By integrating around the following contour

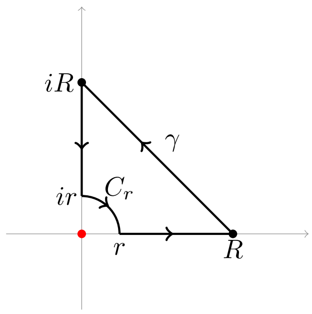


Figure 17: Triangular contour

By the residue theorem

$$\int_{C_r} f(z)\,dz + \int_r^R f(x)\,dx + \int_\gamma f(z)\,dz + \int_{ir}^{iR} f(x)\,dx = 0$$

Taking the integral around the small quarter circle with $r \to 0$

$$\left|\int_{C_r} f(z)\,dz\right| \leq \left|\sqrt{r}\int_0^{\pi/2} e^{-it/2}e^{rie^{it}}\,dt\right| \leq \sqrt{r}\int_0^{\pi/2}\left|e^{-r\sin(t)}\right|\,dt \sim 0$$

On $\gamma(t) = (1-t)R + iRt$ and $0 \leq t \leq 1$

$$\left|\int_\gamma f(z)\,dz\right| = \left|R(i-1)\int_0^1 e^{-1/2\log(R(1-t)+iRt)}e^{i(1-t)R-Rt}\,dt\right| \leq \sqrt{2R}\int_0^1 \frac{e^{-Rt}}{\sqrt[4]{(1-t)^2+t^2}}\,dt$$

Note that on the interval $\sqrt[4]{(1-t)^2+t^2} \geq \sqrt[4]{2}$

$$\left|\int_\gamma f(z)\,dz\right| \leq 2^{1/4}\sqrt{R}\int_0^1 e^{-Rt}\,dt = \frac{2^{1/4}}{\sqrt{R}}\left(1-e^{-R}\right) \sim_\infty 0$$

Finally what is remaining when $r \to 0$ and $R \to \infty$

$$\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\,dx = i\int_0^\infty (ix)^{-1/2}e^{-x}\,dx$$

Note that $i^{-1/2} = e^{-i\pi/4}$

$$\int_0^\infty \frac{e^{ix}}{\sqrt{x}}\,dx = ie^{-i\pi/4}\int_0^\infty x^{-1/2}e^{-x}\,dx = ie^{-i\pi/4}\sqrt{\pi}$$

Using that we have

$$\int_0^\infty \frac{\cos(x)}{\sqrt{x}}\,dx = \int_0^\infty \frac{\sin(x)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

22.9 Residue at infinity

Define the residue at infinite as the following

$$\text{Res}(f,\infty) = -\text{Res}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right),0\right)$$

This is useful especially considering integrals where we wrap the contour around the whole complex plain by adding the point at infinity. Hence if we exclude a certain region of the complex plain and integrate around the whole complex plain (think about it as a sphere) then we can apply the residue theorem using the residue at infinity.

22.9.1 Example

Prove that

$$\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \frac{\pi}{8}$$

proof

Consider the function

$$f(z) = \sqrt{z-z^2} = e^{\frac{1}{2}\log(z-z^2)}$$

Consider the branch cut on the x-axis

$$x(1-x) \geq 0 \implies 0 \leq x \leq 1$$

Let $w = z - z^2$ then

$$\log(w) = \log|w| + i\theta, \ \theta \in [0, 2\pi)$$

Consider the contour where we avoid the two branch points

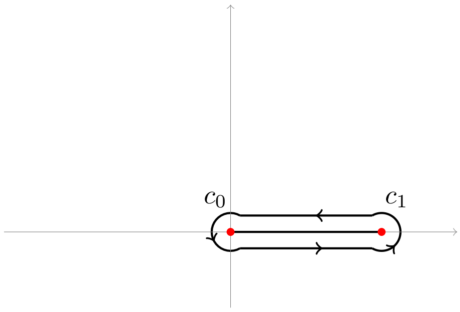


Figure 18: Dumbbell contour

By the residue theorem we have

$$\int_{c_0} f(z)\,dz + \int_{c_1} f(z)\,dz + \int_{\epsilon}^{1-\epsilon} e^{\frac{1}{2}\log|x-x^2|}\,dx - \int_{\epsilon}^{1-\epsilon} e^{\frac{1}{2}\log|x-x^2|+\pi i}\,dx = 2\pi i \operatorname{Res}(f, \infty)$$

Consider the Laurent expansion of

$$\sqrt{z-z^2} = i\sqrt{z^2}\sqrt{1-\frac{1}{z}} = iz\sum_{k=0}^{\infty}\binom{1/2}{k}\left(-\frac{1}{z}\right)^k$$

Hence we deuce that

$$\operatorname{Res}(f, \infty) = -\frac{i}{8}$$

That implies

$$\int_{c_0} f(z)\,dz + \int_{c_1} f(z)\,dz + 2\int_{\epsilon}^{1-\epsilon} \sqrt{x}\sqrt{1-x}\,dx = \frac{\pi}{4}$$

The contours around the branch points go to zero. Finally we get

$$\int_0^1 \sqrt{x}\sqrt{1-x}\,dx = \frac{\pi}{8}$$

22.10 Inverse of Laplace transform

The inverse Laplace transform is defined in the complex plain as

$$f(t) = \mathcal{L}^{-1}\{F\}(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}F(s)ds$$

22.10.1 Example

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(a+t)\Gamma(b-t)s^{-t}\,dt = \frac{\Gamma(a+b)}{(1+s)^{a+b}}s^a$$

proof

Consider the following function

$$f(z) = \Gamma(z+a)\Gamma(b-z)s^{-z}$$

Suppose that $a, b \in \mathbb{R}$ and $a < b$. Note that the Gamma function has a pole of order 1 at each non-positive integer where we have

$$\operatorname{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}$$

The function f has poles at the following points

$$-n-a, -(n-1)-a, \cdots, -a, b, b+1, \cdots, b+n$$

Notice that the function f is analytic on the region $-a < \operatorname{Re}(z) < b$, hence consider the contour in Figure 19.

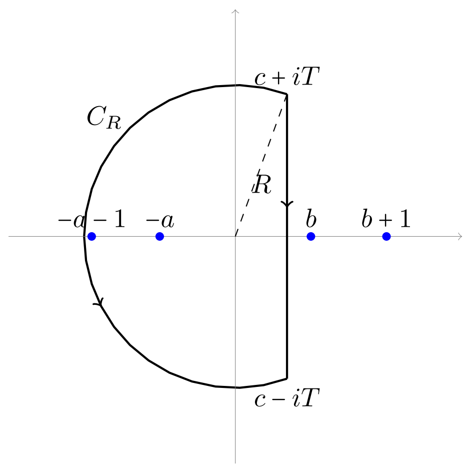


Figure 19: Bromwich contour

By the residue theorem

$$\int_{C_R} f(z)\,dz + \int_{c-iT}^{c+iT} f(z)\,dz = 2\pi i \sum_{k=0}^{n} \operatorname*{Res}_{z=-a-k} f(z)$$

By taking the limit → ∞ we have

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz + \int_{c-i\infty}^{c+i\infty} f(t)\,dt = 2\pi i \sum_{k=0}^{\infty} \operatorname*{Res}_{z=-a-k} f(z)$$

Note that

$$\operatorname*{Res}_{z=-a-k} \Gamma(z+a)\Gamma(b-z)s^{-z} = \Gamma(b+a+k) \lim_{z\to-a-k} \Gamma(-k)s^{a+k} = \frac{(-1)^k\Gamma(a+b+k)}{\Gamma(k+1)} s^{a+k}$$

Note that

$$s^a \sum_{k=0}^{\infty} \frac{\Gamma(a+b+k)}{\Gamma(k+1)}(-s)^k = s^a\Gamma(a+b) \sum_{k=0}^{\infty} (a+b)_k \frac{(-s)^k}{k!} = {}_2F_1(1;1,a+b,-s)$$

By definition of the Hypergeometric funciton

$$s^a\Gamma(a+b) \sum_{k=0}^{\infty} (a+b)_k \frac{(-s)^k}{k!} = s^a\Gamma(a+b){}_2F_1(a+b,1;1,-s) = \frac{\Gamma(a+b)}{(1+s)^{a+b}} s^a$$

Also notice that

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

So we deduce that

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(a+t)\Gamma(b-t)s^{-t}\,dt = \frac{\Gamma(a+b)}{(1+s)^{a+b}} s^a$$

22.11 Infinite sums

If we have an infinite number of poles then by taking a contour that covers all of them we have an infinite number of residues that create an infinite series.

22.11.1 Example

Prove that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$$

proof

Consider the function

$$f(z) = \frac{(\psi(-z)+\gamma)^2}{z^2}$$

Note that f has poles at non-negative integers

By integration around a large circle $|z| = \rho$

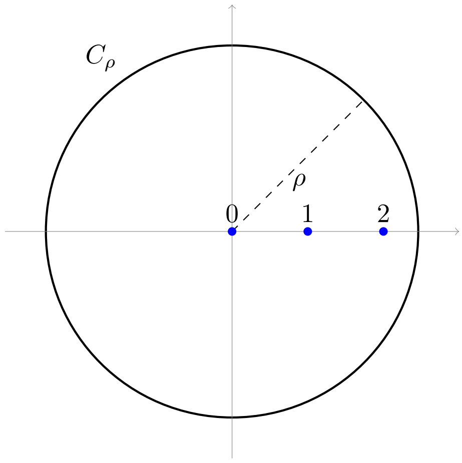


Figure 20: Contour $|z| = \rho$

Note that

$$\oint f(z)\,dz = 2\pi i(\text{Res}(f,0) + \sum_{n=1}^{\infty} \text{Res}(f,n))$$

The integration around the circle

$$\left| \int_{|z|=\rho} \frac{(\psi(-z)+\gamma)^2}{z^2}\,dz \right| \le 2\pi \frac{(|\psi(-z)| + |\gamma|)^2}{\rho}$$

As $|z| \to \infty$ we have $\psi(-z) \sim \log(-z)$ and the principle logarithm

$$\left| \int_{|z|=\rho} \frac{(\psi(-z)+\gamma)^2}{z^2}\,dz \right| \le 2\pi \frac{(|\log(-z)| + |\gamma|)^2}{\rho} \le 2\pi \frac{(|\log(\rho)| + 2\pi + |\gamma|)^2}{\rho} \sim_\infty 0$$

We deduce that

$$2\pi i(\text{Res}(f,0) + \sum_{n=1}^{\infty} \text{Res}(f,n)) = 0$$

By expansion near $z = 0$

$$\frac{(\psi(-z)+\gamma)^2}{z^2} \approx \frac{1}{z^2} - 2\frac{\zeta(2)}{z^2} - 2\frac{\zeta(3)}{z}$$

Which implies that

$$\operatorname{Res}(f,0) = -2\zeta(3)$$

For the residues at non-negative integers n, the expansions

$$\frac{1}{z^2} = \sum_{j=0}^{\infty} (-1)^j \binom{j+1}{1} \frac{(z-n)^j}{n^{2+j}}$$

$$\psi(-z) + \gamma = \frac{1}{z-n} + H_n + \sum_{k=1}^{\infty} ((-1)^k H_n^{(k+1)} - \zeta(k+1))(z-n)^k$$

This implies that

$$(\psi(-z) + \gamma)^2 \approx \frac{1}{(z-n)^2} + 2\frac{H_n}{(z-n)}$$

This implies the residue

$$\operatorname{Res}(f,n) = 2\frac{H_n}{n^2} - \frac{2}{n^3}$$

We then deduce that

$$2\sum_{n=1}^{\infty} \left[\frac{H_n}{n^2} - \frac{1}{n^3}\right] - 2\zeta(3) = 0$$

Finally we get

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$$

References

[1] Leonard Lewin. *Polylogarithms and associated functions*. New York, 1981.

[2] Marko Petkovsek, Herbert Wilf, Dorom Zeilbegreger. *A=B*. Philadelphia, 1997.

[3] Pedro Freitas. *Integrals of polylogarithmic functions, recurrence relations and associated Euler sums*. 2004.

[4] Gradshteyn, Ryhik. *Edited by* Alan Jeffrey, Daniel Zillinger. *Table of integrals, series and products*. 2007.

[5] Habib Muzaffiar, Kenneth Williams. *Evaluation of complete elleptic integrals of the first kind at singular moduli*. 2006.

[6] Stefan Boettner, Victor Moll. *The integrals in Gradshteyn and Ryzhik, Part 16: Complete elliptic integrals*. 2010.

[7] *Factorial, Gamma and Beta function* `http://mhtlab.uwaterloo.ca/courses/me755/web_chap1.pdf`

[8] Keith Conrad. *Differentiation under the integral sign*.

[9] `http://www.mymathforum.com`

[10] `http://www.mathhelpboards.com`

[11] `http://www.mathstackexchange.com`

[12] `http://www.integralsandseries.prophpbb.com`

[13] `http://www.wikipedia.org`

[14] `http://mathworld.wolfram.com`

[15] `http://dlmf.nist.gov/`

[16] Dennis G. Zill, Patrick D. Shanahan. *A First Course in Complex Analysis with Applications*. 2003.

[17] G. N Watson, M.A. *Complex integration and Cauchy's theorem*. 1914.

[18] Ravi P. Agarwal, Kanishka Perera, Sandra Pinelas. *An Introduction to Complex Analysis*. 2011.