Advanced Integration Techniques

$\int dx$

Advanced approaches for solving many complex integrals using special functions and some transformations

Second Version

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Contents

Acknowledgement

I want to offer my sincerest gratitude to all those who supported me during my journey to finish this book. Especially my parents, sisters and friends who supported the idea of this book. I also want to thank my Math teachers at King Fahd University because I wouldn't be able to learn the advanced without having knowledge of the elementary. I also want to extend my thanks to all my friends on the different math forums like MMF, MHB and stack exchange without them I wouldn't be learning any thing.

Reviewers

A special thank for **Mohammad Nather Shaaban** for reviewing some parts of the book.

What is new?

Basically, there are 5 new functions added to the book. The Cosine and Sine Integral functions, the Clausen functions, the logarithm integral function and the Barnes G function. The structure of the book is basically the same. Many typos and computation mistakes were corrected.

The future work

I have a plan to add many other sections. Basically I'll try to focus on transformations like Mellin and fourier transforms. Also many other functions like the Jacobi theta function and q-series. Also I am thinking of adding a long section about contour integration.

Introduction

This book is a summary of working on advanced integrations for around four years. It collects many examples that I gathered during that period. The approaches taken to solve the integrals aren't necessarily the only and best methods but they are offered for the sake of explaining the topic. Most of the content of this book I already wrote on mathhelpboards.com during the past three years but I thought that publishing it using a pdf would be easier to read and distribute. The motivation behind this book is to allow those who are interested in solving complicated integrals to be able to use the different methods to solve them efficiently. When I started learning about these techniques I would suffer to get enough information about all the required approaches so I tried to collect every thing in just one book. You are free to distribute this book and use any of the methods to solve the integrals or use the same techniques. The methods used are not necessarily new or ground-breaking but as I said they introduce the concept as easy as possible.

To follow this book you have to be know the basic integration techniques like integration by parts, by substitution and by partial fractions. I don't assume that the readers know any other stuff from any other topics or advanced courses from mathematics. Usually the details that require deep knowledge of analysis or advanced topics are left or just touched upon lightly to give the reader some hints but not going into details.

After reading this book you should be able to solve many advanced integrals that you might face in engineering courses. I hope you enjoy reading this book and if you have any suggestions, comments or correction I will be happy to recieve them through my email `mailto:alyafey22@gmail.com` or this email `mailto:alyafey_22@hotmail.com`. Also I am avilable as a staff member at `http://www.mathhelpboards.com` if you have some questions that I could reply to you directly using Latex.

1 Differentiation under the integral sign

This is one of the most commonly used techniques to solve a numerous number of questions.
Assume that we have the following function of two variables

$$\int_a^b f(x,y)\,dx$$

Then we can differentiate with respect to y provided that f is continuous and has a partial continuous derivative on a chosen interval

$$F'(y) = \int_a^b f_y(x,y)\,dx$$

Now using this in many problems is not that clear you have to think a lot to get the required answer because many integrals are usually in one variable so you need to introduce the second variable and assume it is a function of two variables.

1.1 Example

Assume we want to solve the following integral

$$\int_0^1 \frac{x^2-1}{\log(x)}\,dx$$

That seems very difficult to solve but using this technique we can solve it easily. The crux move is to decide where to put the second variable! So the problem with the integral is that we have a logarithm in the denominator which makes the problem so difficult to tackle! Remember that we can get a natural logarithm if we differentiate exponential functions i.e $F(a) = 2^a \Rightarrow F'(a) = \log(a)\cdot 2^a$
Applying this to our problem

$$F(a) = \int_0^1 \frac{x^a-1}{\log(x)}\,dx$$

Now we take the partial derivative with respect to a

$$F'(a) = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a-1}{\log(x)}\right)\,dx = \int_0^1 x^a\,dx = \frac{1}{a+1}$$

Integrate with respect to a

$$F(a) = \log(a+1) + C$$

To find the value of the constant put $a = 0$

$$F(0) = \log(1) + C \implies C = 0$$

This implies that

$$\int_0^1 \frac{x^a - 1}{\log(x)}\, dx = \log(a+1)$$

By this powerful method we were not only able to solve the integral we also found a general formula for some a where the function is differentiable in the second variable.

To solve our original integral put $a = 2$

$$\int_0^1 \frac{x^2 - 1}{\log(x)}\, dx = \log(2+1) = \log(3)$$

1.2 Example

Find the following integral

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan x} dx$$

So where do we put the variable a here? that doesn't seem to be straight forward , how do we proceed ?

Let us try the following

$$F(a) = \int_0^{\frac{\pi}{2}} \frac{\arctan(a\tan(x))}{\tan(x)} dx$$

Now differentiate with respect to a

$$F'(a) = \int_0^{\frac{\pi}{2}} \frac{1}{1 + (a\tan(x))^2} dx$$

It can be proved that

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + (a\tan(x))^2} dx = \frac{\pi}{2(1+a)}$$

Now Integrate both sides

$$F(a) = \frac{\pi}{2}\log(1+a) + C$$

Substitute $a = 0$ to find $C = 0$

$$\int_0^{\frac{\pi}{2}} \frac{\arctan(a\tan(x))}{\tan(x)}\,dx = \frac{\pi}{2}\log(1+a)$$

Put $a = 1$ in order to get our original integral

$$\int_0^{\frac{\pi}{2}} \frac{x}{\tan(x)}\,dx = \frac{\pi}{2}\log(2)$$

1.3 Example

$$\int_0^{\infty} \frac{\sin(x)}{x}\,dx$$

This problem can be solved by many ways , but here we will try to solve it by differentiation. So as I showed in the previous examples it is generally not easy to find the function to differentiate. Actually this step might require trial and error techniques until we get the desired result, so don't just give up if an approach doesn't work!

Let us try this one

$$F(a) = \int_0^{\infty} \frac{\sin(ax)}{x}\,dx$$

If we differentiated with respect to a we would get the following

$$F'(a) = \int_0^{\infty} \cos(ax)\,dx$$

But unfortunately this integral doesn't converge, so this is not the correct one. Actually, the previous theorem will not work here because the integral is improper.

So let us try the following

$$F(a) = \int_0^{\infty} \frac{\sin(x)e^{-ax}}{x}\,dx$$

Take the derivative

$$F'(a) = -\int_0^{\infty} \sin(x)e^{-ax}\,dx$$

Use integration by parts twice

$$F'(a) = -\int_0^{\infty} \sin(x)e^{-ax}\,dx = \frac{-1}{a^2+1}$$

Integrate both sides

$$F(a) = -\arctan(a) + C$$

To find the value of the constant take the limit as a grows large

$$C = \lim_{a\to\infty} F(a) + \arctan(a) = \frac{\pi}{2}$$

So we get our $F(a)$ as the following

$$F(a) = -\arctan(a) + \frac{\pi}{2}$$

For a = 0 we have

$$\int_0^\infty \frac{\sin(x)}{x} = \frac{\pi}{2}$$

2 Laplace Transform

2.1 Basic Introduction

Laplace transform is a powerful integral transform. It can be used in many applications. For example, it can be used to solve Differential Equations and its rules can be used to solve integration problems.
The basic definition of Laplace transform

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

This integral will converge when

$$\mathrm{Re}(s) > a \ , \ |f(t)| \le M e^{at}$$

Let us see the Laplace transform for some functions

2.1.1 Example

Find the Laplace transform of the following functions

1. $f(t) = 1$

$$F(s) = \int_0^\infty e^{-st}\, dt = \frac{1}{s}$$

2. For $f(t) = t^n$ where $n \ge 0$

   We can prove using integration by parts

$$F(s) = \int_0^\infty e^{-st} t^n\, dt = \frac{n!}{s^{n+1}}$$

3. For the geometric function $f(t) = \cos(at)$, Use integration by parts

$$F(s) = \int_0^\infty e^{-st} \cos(at)\, dt = \frac{s}{s^2 + a^2}$$

2.2 Example

Find the following integral

$$\int_0^\infty e^{-2t} t^3 \, dt$$

We can directly use the formula in the previous example

$$\int_0^\infty e^{-st} t^n \, dt = \frac{n!}{s^{n+1}}$$

Here we have $s = 2$ and $n = 3$

$$\int_0^\infty e^{-2t} t^3 \, dt = \frac{3!}{2^{3+1}} = \frac{3}{8}$$

2.3 Convolution

Define the following integral

$$(f * g)(t) = \int_0^t f(s) g(t - s) \, ds$$

Then we have the following

$$\mathcal{L}\left((f * g)(t)\right) = \mathcal{L}(f(t))\mathcal{L}(g(t))$$

2.4 Inverse Laplace transform

So, basically you are given $F(s)$ and we want to get $f(t)$ this is denoted by

$$\mathcal{L}(f(t)) = F(s) \implies \mathcal{L}^{-1}(F(s)) = f(t)$$

2.4.1 Example

Find the inverse Laplace transform of

1. $F(s) = \frac{1}{s^3}$

   We use the results applied previously

$$\mathcal{L}(t^2) = \frac{2!}{s^3} \Rightarrow \frac{1}{2}\mathcal{L}(t^2) = \frac{1}{s^3}$$

Now take the inverse to both sides

$$\frac{t^2}{2} = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right)$$

2. $F(s) = \frac{s}{s^2+4}$

   we can use the Laplace of cosine to deduce

$$\cos(2t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)$$

Exercises

Find the Laplace transform

$$\sin(at)$$

Find the inverse Laplace

$$\frac{1}{s^{n+1}}$$

2.5 Interesting results

2.5.1 Example

Prove the following

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

β is the Beta function and Γ is the Gamma function. We will take enough time and examples to explain both functions in the next sections.

proof

We need convolution rule we described earlier

Let us choose some functions f and g

$$f(t) = t^x \ , \ g(t) = t^y$$

Hence we get

$$(t^x * t^y) = \int_0^t s^x (t-s)^y \, ds$$

So by the convolution rule we have the following

$$\mathcal{L}\left(t^x * t^y\right) = \mathcal{L}(t^x)\mathcal{L}(t^y)$$

We can now use the Laplace of the power

$$\mathcal{L}\left(t^x * t^y\right) = \frac{x! \cdot y!}{s^{x+y+2}}$$

Notice that we need to find the inverse of Laplace $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\left(\mathcal{L}(t^x * t^y)\right) = \mathcal{L}^{-1}\left(\frac{x! \cdot y!}{s^{x+y+2}}\right) = t^{x+y+1}\frac{x! \cdot y!}{(x+y+1)!}$$

So we have the following

$$\left(t^x * t^y\right) = t^{x+y+1}\frac{x! \cdot y!}{(x+y+1)!}$$

By definition we have

$$t^{x+y+1}\frac{x! \cdot y!}{(x+y+1)!} = \int_0^t s^x (t-s)^y \, ds$$

Now put $t = 1$ we get

$$\frac{x! \cdot y!}{(x+y+1)!} = \int_0^1 s^x (1-s)^y \, ds$$

By using that $n! = \Gamma(n+1)$ we deduce that

$$\int_0^1 s^x (1-s)^y \, ds = \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$$

which can be written as

$$\int_0^1 s^{x-1} (1-s)^{y-1} \, ds = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

2.5.2 Example

Prove the following

$$\int_0^\infty \frac{f(t)}{t} \, dt = \int_0^\infty \mathcal{L}(f(t)) \, ds$$

proof

we know from the definition

$$\int_0^\infty \mathcal{L}(f(t))\,ds = \int_0^\infty \left(\int_0^\infty e^{-st} f(t)\,dt\right) ds$$

Now by the Fubini theorem we can rearrange the double integral

$$\int_0^\infty f(t)\left(\int_0^\infty e^{-st}\,ds\right) dt$$

The integral inside the parenthesis

$$\int_0^\infty e^{-st}\,ds = \frac{1}{t}$$

Now substitute this value in the integral

$$\int_0^\infty \frac{f(t)}{t}\,dt$$

2.5.3 Example

Find the following integral

$$\int_0^\infty \frac{\sin(t)}{t}\,dt$$

This is not the first time we see this integral and not the last . We have seen that we can find it using differentiation under the integral sign.

Let us use the previous example

$$\int_0^\infty \frac{\sin(t)}{t}\,dt = \int_0^\infty \mathcal{L}(\sin(t))\,ds$$

We can prove that

$$\mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$$

Substitute in our integral

$$\int_0^\infty \frac{ds}{1+s^2} = \tan^{-1}(s)|_{s=\infty} - \tan^{-1}(s)|_{s=0} = \frac{\pi}{2}$$

3 Gamma Function

The gamma function is used to solve many interesting integrals, here we try to define some basic properties, prove some of them and take some examples.

3.1 Definition

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^x \, dt$$

For the first glance that just looks like the Laplace Transform, actually they are closely related.
So let us for simplicity assume that $x = n$ where $n \geq 0$ (is an integer)

$$\Gamma(n+1) = \int_0^\infty e^{-t}\, t^n \, dt$$

We can use the Laplace transform

$$\int_0^\infty e^{-t}\, t^n = \frac{n!}{s^{n+1}}|_{s=1} = n!$$

So we see that there is a relation between the gamma function and the factorial. We will assume for the time being that the gamma function is defined as the following

$$n! = \Gamma(n+1)$$

This definition is somehow limited but it will be soon replaced by a stronger one.

3.2 Example

Find the following integrals

$$\int_0^\infty e^{-t} t^4 \, dt$$

By definition this can be replaced by

$$\int_0^\infty e^{-t} t^4 \, dt = \Gamma(4+1) = 4! = 24$$

3.3 Example

Solving the following integrals

1. $$\int_0^\infty e^{-t^2}\, t\, dt$$

   We need a substitution before we go ahead, so let us start by putting $x = t^2$ so the integral becomes

   $$\frac{1}{2}\int_0^\infty e^{-x} x^{\frac{1}{2}} \cdot x^{-\frac{1}{2}}\, dt = \frac{1}{2}\Gamma(1+0) = \frac{1}{2}$$

2. $$\int_0^1 \log(t)\, t^2\, dt$$

   we use the substitution $t = e^{-\frac{x}{2}}$

   $$\int_0^1 \log(t)\, t^2\, dt = -\frac{1}{4}\int_0^\infty e^{-\frac{3x}{2}} \cdot x\, dx$$

   Using another substitution $t = \frac{3x}{2}$

   $$\frac{-1}{9}\int_0^\infty e^{-x}\, x\, dx = \frac{-\Gamma(2)}{9} = \frac{-1}{9}$$

It is an important thing to get used to the symbol Γ. I am sure that you are saying that this seems elementary, but my main aim here is to let you practice the new symbol and get used to solving some problems using it.

3.4 Exercises

Prove that

$$\frac{\Gamma(5)\cdot\Gamma(2)}{\Gamma(7)} = \frac{1}{30}$$

Find the following integral

$$\int_0^\infty e^{-\frac{1}{60}t}\, t^{20}\, dt$$

3.5 Extension

For simplicity we assumed that the gamma function only works for positive integers. This definition was so helpful as we assumed the relation between gamma and factorial. Actually, this restricts the gamma function, we want to exploit the real strength of this function. Hence, we must extend the gamma function to work for all real numbers except for some values. Actually we will see soon that we can extend it to work for all complex numbers except where the function has poles.

3.5.1 Theorem

Using the integral representation we can extend the gamma function to $x > -1$.

proof

We need only consider the case when $-1 < x < 0$.

Near infinity we have the following

$$\left|\int_0^{\infty} e^{-t}\, t^x\, dt\right| \le \int_{\epsilon}^{\infty} e^{-t} dt < \infty$$

Near zero when $x = -z$ we have the following

$$\left|\int_0^{\epsilon} \frac{e^{-t}}{t^z}\, dt\right| \sim \int_0^{\epsilon} \frac{1}{t^z} dt < \infty$$

3.5.2 Reduction formula

$$\Gamma(x+1) = x\Gamma(x)$$

This can be proved through integration by parts for $x > 0$. Actually this representation allows us to extend the gamma function for all real numbers for non-negative integers. In terms of complex analysis this function is analytic except at non-positive integers where it has poles.

3.6 Other Representations

3.6.1 Euler Representation

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z}{z} \prod_{k=1}^{n} \frac{k}{k+z} = \frac{1}{z} \prod_{k=1}^{\infty} \frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}}$$

proof

Note that

$$\Gamma(z+n+1) = \Gamma(z+1) \prod_{k=1}^{n} (k+z)$$

Which indicates that

$$\prod_{k=1}^{n}(k+z)=\frac{\Gamma(z+n+1)}{z\Gamma(z)}$$

Also note that

$$\prod_{k=1}^{n}k=n!$$

Hence we have

$$\lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{k+z}=\Gamma(z)\lim_{n\to\infty}\frac{n^z\times n!}{\Gamma(z+n+1)}$$

Hence we must show that

$$\lim_{n\to\infty}\frac{n^z\times n!}{\Gamma(z+n+1)}=1$$

Note that by Stirling formula

$$\Gamma(z+n+1)\sim\sqrt{2\pi}(n+z)^{n+z+1/2}e^{-(n+z)}$$

and

$$n!\sim\sqrt{2\pi}n^{n+1/2}e^{-n}$$

Hence we have by

$$\lim_{n\to\infty}\frac{n^z\times(\sqrt{2\pi}n^{n+1/2}e^{-n})}{\sqrt{2\pi}(n+z)^{n+z+1/2}e^{-(n+z)}}=\lim_{n\to\infty}\frac{n^n}{(n+z)^n e^{-z}}=1$$

Note that

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=e^z$$

To prove the other product formula note that

$$\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z=\frac{\prod_{k=1}^{n}(1+k)^z}{\prod_{k=1}^{n}k^z}=(n+1)^z\sim n^z$$

Hence we deduce

$$\lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{k+z}=\lim_{n\to\infty}\frac{\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z}{z}\prod_{k=1}^{n}\frac{1}{1+\frac{z}{k}}=\frac{1}{z}\prod_{k=1}^{\infty}\frac{\left(1+\frac{1}{k}\right)^z}{1+\frac{z}{k}}$$

3.6.2 Example

Prove that

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \prod_{k=0}^{\infty}\left[\left(1+\frac{z}{x+k}\right)\left(1-\frac{z}{y+k}\right)\right]$$

proof

Start by

$$\Gamma(z) = \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{k+z}$$

We have

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \lim_{n\to\infty}\frac{\left(\frac{n^x}{x}\prod_{k=1}^{n}\frac{k}{k+x}\right)\left(\frac{n^y}{y}\prod_{k=1}^{n}\frac{k}{k+y}\right)}{\left(\frac{n^{x+z}}{x+z}\prod_{k=1}^{n}\frac{k}{k+x+z}\right)\left(\frac{n^{y-z}}{y-z}\prod_{k=1}^{n}\frac{k}{k+y-z}\right)}$$

By simplifications we have

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \lim_{n\to\infty}\frac{(x+z)(y-z)}{xy}\prod_{k=1}^{n}\frac{(k+x+z)(k+y-z)}{(k+x)(k+y)}$$

This simplifies to

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+z)\Gamma(y-z)} = \prod_{k=0}^{\infty}\frac{(k+x+z)(k+y-z)}{(k+x)(k+y)} = \prod_{k=0}^{\infty}\left[\left(1+\frac{z}{x+k}\right)\left(1-\frac{z}{y+k}\right)\right]$$

3.6.3 Weierstrass Representation

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n}$$

where γ is the Euler constant

proof

Take logarithm to the Euler representation

$$\log z\Gamma(z) = \lim_{n\to\infty} z\sum_{k=1}^{n}\left(\log(1+k)-\log(k)\right)-\sum_{k=1}^{n}\log\left(1+\frac{z}{k}\right)$$

Note the alternating sum

$$\sum_{k=1}^{n}\left(\log(1+k)-\log(k)\right) = \log(n+1)$$

Hence we have

$$\log z\Gamma(z) = \lim_{n\to\infty} z\log(n+1) - \sum_{k=1}^{n} \log\left(1+\frac{z}{k}\right)$$

Now we can use the harmonic numbers

$$H_n = \sum_{k=1}^{n} \frac{1}{n}$$

Add and subtract zH_{n+1}

$$\log z\Gamma(z) = \lim_{n\to\infty} z\log(n+1) - zH_{n+1} + \sum_{k=1}^{n}\left[\log\left(1+\frac{z}{k}\right)^{-1} + \frac{z}{k}\right] + \frac{z}{n+1}$$

The last term goes to zero and by definition we have the Euler constant

$$\gamma = \lim_{n\to\infty} H_n - \log(n)$$

Hence the first term is the Euler constant

$$\log z\Gamma(z) = -z\gamma + \sum_{k=1}^{\infty} \log\left(1+\frac{z}{k}\right)^{-1} + \frac{z}{k}$$

By taking the exponent of both sides

$$z\Gamma(z) = e^{-\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}}$$

3.7 Laurent expansion

$$\Gamma(z) = \frac{1}{z} - \gamma + \frac{1}{2!}(\gamma^2 + \zeta(2))z + O(z^2)$$

proof

Note that $f(z) = \Gamma(z+1)$ has a Maclurain expansion near 0

$$\Gamma(z+1) = \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} z^k$$

For the first term

$$f(0) = \Gamma(1+0) = 1$$

For the second term

$$\frac{f'(0)}{1!} = \Gamma'(1)$$

To find the derivative, note that by the Weierstrass representation

$$\log\Gamma(z) \;= -\gamma z - \log(z) + \sum_{n=1}^{\infty} \log\left(1+\frac{z}{n}\right)^{-1} + \frac{z}{n}$$

By taking the derivaive we have

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty}\frac{1}{k(z+k)}$$

Hence we have

$$\Gamma'(1) = -\gamma - 1 + \sum_{k=1}^{\infty}\frac{1}{k(1+k)} = -\gamma - 1 + 1 = -\gamma$$

For the third term

$$\frac{f''(0)}{2!} = \frac{\Gamma''(1)}{2}$$

Taking the second derivative

$$\frac{\Gamma''(z)\Gamma(z) - (\Gamma'(z))^2}{\Gamma^2(z)} = \frac{1}{z^2} + \sum_{k=1}^{\infty}\frac{1}{(z+k)^2}$$

Which indicates that

$$\Gamma''(1) = (\Gamma'(z))^2 + 1 + \sum_{k=1}^{\infty}\frac{1}{(1+k)^2} = \gamma^2 + \zeta(2)$$

Hence we deduce that

$$\Gamma(z+1) = 1 - \gamma z + \frac{1}{2!}(\gamma^2 + \zeta(2))z^2 + O(z^3)$$

Dividing by z we get our result.

3.8 Example

Find the integral

$$\int_0^{\infty}\frac{e^{-t}}{\sqrt{t}}\,dt$$

Now according to our definition this is equal to $\Gamma\left(\frac{1}{2}\right)$ but this value can be represented using elementary functions as follows

Let us first make a substitution $\sqrt{t} = x$

$$2\int_0^{\infty} e^{-x^2}\,dx$$

Now to find this integral we need to do a simple trick, start by the following

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\cdot\left(\int_0^\infty e^{-x^2}\,dx\right)$$

Since x is a dummy variable we can put

$$\left(\int_0^\infty e^{-x^2}\,dx\right)\cdot\left(\int_0^\infty e^{-y^2}\,dy\right)$$

Now since they are two independent variables we can do the following

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$

Now by polar substitution we get

$$\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

The inner integral is $\frac{1}{2}$, hence we get

$$\int_0^{\frac{\pi}{2}}\frac{1}{2}\,d\theta = \frac{\pi}{4}$$

So we have

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

Take the square root to both sides

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$2\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}$$

So we have our result

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

3.9 More values

We can use the reduction formula and the value of $\Gamma(1/2)$ to deduce other values. Assume that we want to find

$$\Gamma\left(\frac{3}{2}\right)$$

If we used this property we get

$$\Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

Not all the time the result will be reduced to a simpler form as the previous example. For example we don't know how to express $\Gamma(\frac{1}{4})$ in a simpler form but we can approximate its value

$$\Gamma\left(\frac{1}{4}\right) \approx 3.6256\cdots$$

Hence we just solve some integrals in terms of gamma function since we don't know a simpler form. For example solve the integral

$$\int_0^\infty e^{-t} t^{\frac{1}{4}}\, dt$$

we know by definition of gamma function that this reduces to

$$\int_0^\infty e^{-t} t^{\frac{1}{4}}\, dt = \Gamma\left(\frac{5}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)}{4}$$

We have seen that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ but what about $\Gamma\left(\frac{-1}{2}\right)$?
By the reduction formula

$$\Gamma\left(1-\frac{1}{2}\right) = \frac{-1}{2}\Gamma\left(\frac{-1}{2}\right)$$

so we have that

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

Then we can prove that any fraction where the denominator equals to 2 and the numerator is odd can be reduced into

$$\Gamma\left(\frac{2n+1}{2}\right) = C\,\Gamma\left(\frac{1}{2}\right)\ ,\ C \in \mathbb{Q}\ ,\ n \in \mathbb{Z}$$

3.10 Legendre Duplication Formula

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi}$$

proof

For the proof we use induction by assuming $n \geq 0$. If $n = 0$ we have our basic identity

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Now we need to prove that

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \implies \Gamma\left(\frac{1}{2}+n+1\right) = \frac{(2n+2)!}{4^{n+1}(n+1)!}\sqrt{\pi}$$

Now we use the reduction formula

$$\Gamma\left(\frac{1}{2}+n+1\right) = \frac{1+2n}{2}\Gamma\left(\frac{1}{2}+n\right)$$

By the inductive step we have

$$\frac{1+2n}{2}\Gamma\left(\frac{1}{2}+n\right) = \frac{1+2n}{2}\cdot\frac{(2n)!}{4^n n!}\sqrt{\pi}$$

We can multiply and divide by $2n+2$

$$\frac{1+2n}{2}\cdot\frac{(2n)!}{4^n n!}\sqrt{\pi}\cdot\frac{2n+2}{2n+2} = \frac{(2n+2)!}{4^{n+1}(n+1)!}\sqrt{\pi}$$

3.11 Example

$$\int_0^\infty \frac{e^{-t}\cosh(a\sqrt{t})}{\sqrt{t}}dt$$

we have a hyperbolic function

We know that we can expand $\cosh$ using power series

$$\cosh(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

Let $x = a\sqrt{t}$

$$\cosh(a\sqrt{t}) = \sum_{n=0}^{\infty}\frac{a^{2n}\cdot t^n}{(2n)!}$$

Substituting back in the integral we have

$$\int_0^\infty e^{-t} \sum_{n=0}^\infty \frac{a^{2n} \cdot t^n}{(2n)!\,\sqrt{t}} dt$$

Now since the series is always positive we can swap the integral and the series

$$\sum_{n=0}^\infty \frac{a^{2n}}{(2n)!} \left[\int_0^\infty e^{-t} t^{n-\frac{1}{2}}\, dt\right]$$

Hence we have by using the gamma function

$$\sum_{n=0}^\infty \frac{a^{2n}\,\Gamma\left(\frac{1}{2}+n\right)}{(2n!)}$$

Using LDF (Legendre Duplication Formula) we get

$$\sum_{n=0}^\infty \frac{a^{2n}}{(2n!)} \left(\frac{(2n)!}{4^n n!}\sqrt{\pi}\right)$$

By further simplification

$$\sqrt{\pi}\, \sum_{n=0}^\infty \frac{a^{2n}}{4^n\, n!}$$

Now that looks familiar since we know that

$$\sum_{n=0}^\infty \frac{z^n}{n!} = e^z$$

Putting $z = \frac{a^2}{4}$ and multiplying by $\sqrt{\pi}$ we get

$$\sqrt{\pi} \sum_{n=0}^\infty \frac{\left(\frac{a^2}{4}\right)^n}{n!} = \sqrt{\pi}\, e^{\frac{a^2}{4}}$$

So we have finally that

$$\int_0^\infty \frac{e^{-t}\cosh(a\sqrt{t})}{\sqrt{t}} dt \;=\; \sqrt{\pi} e^{\frac{a^2}{4}}$$

3.12 Euler's Reflection Formula

$$\Gamma\left(z\right)\Gamma\left(1-z\right) = \frac{\pi}{\sin\left(\pi z\right)} \quad , \forall z \notin \mathbb{Z}$$

proof

We have to use the sine infinite product formula

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} \prod_{n=1}^\infty \left(1 - \frac{z^2}{n^2}\right)^{-1}$$

Now we start by noting that

$$\Gamma(z)\Gamma(1-z) = -z\Gamma(z)\Gamma(-z)$$

Now using the Weierstrass formula we have

$$-z\Gamma(z)\Gamma(-z) = -z \cdot \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \cdot \frac{e^{\gamma z}}{-z} \prod_{n=1}^{\infty} \left(1-\frac{z}{n}\right)^{-1} e^{-z/n}$$

This simplifies to

$$\frac{1}{z} \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right)^{-1} = \frac{\pi}{\sin(\pi z)}$$

3.13 Example

Find the following

1.
$$\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

   The first example we can write

$$\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)$$

   Now by ERF (Euler reflection formula) we have the following

$$\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}\,\pi$$

2.
$$\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(\frac{1-i}{2}\right)$$

   Using the same idea for the second one

$$\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(1-\frac{1+i}{2}\right)$$

   This expression simplifies to

$$\frac{\pi}{\sin\left(\frac{\pi(1+i)}{2}\right)} = \frac{\pi}{\cos\left(\frac{i\pi}{2}\right)}$$

By geometry to hyperbolic conversions we get

$$\frac{\pi}{\cosh\left(\frac{\pi}{2}\right)} = \pi \operatorname{sech}\left(\frac{\pi}{2}\right)$$

3.14 Example

Find the integral

$$\int_a^{a+1} \log\Gamma(x)dx$$

Let the following

$$f(a) = \int_a^{a+1} \log\Gamma(x)dx$$

Differentiate both sides

$$f'(a) = \log\Gamma(1+a) - \log\Gamma(a) = \log(a)$$

Integrate both sides

$$f(a) = a\log(a) - a + C$$

Let $a \to 0$

We have

$$C = \int_0^1 \log\Gamma(x)\,dx$$

By the reflection formula

$$\int_0^1 \log\Gamma(x)\,dx = \int_0^1 \log\pi\,dx - \int_0^1 \log\sin(\pi x)\,dx - \int_0^1 \log\Gamma(1-x)\,dx$$

Which implies that

$$2\int_0^1 \log\Gamma(x)\,dx = \int_0^1 \log\pi\,dx - \int_0^1 \log\sin(\pi x)\,dx = \log(2\pi) - \int_0^1 \log|2\sin(\pi x)|\,dx$$

Note that this is the Clausen Integral

$$\int_0^1 \log|2\sin(\pi x)|\,dx = \frac{2}{\pi}\int_0^{2\pi} \log|2\sin(x/2)|\,dx = \frac{2}{\pi}\mathrm{cl}_2(2\pi) = 0$$

Hence we finalize by

$$\int_0^1 \log \Gamma(x) \, dx = \frac{1}{2} \log(2\pi)$$

4 Beta Function

4.1 Representations

4.1.1 First integral formula

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = B(x,y)$$

It is related to the gamma function through the identity

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

We have proved this identity earlier when we discussed convolution.

We shall realize the symmetry of beta function that is to say

$$\beta(x,y) = \beta(y,x)$$

Beta function has many other representations all can be deduced through substitutions

4.1.2 Second integral formula

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt$$

4.1.3 Geometric representation

$$B(x,y) = 2\int_0^{\frac{\pi}{2}} \cos^{2x-1}(t)\sin^{2y-1}(t)\,dt$$

The proofs are left to the reader as practice.

4.2 Example

Prove the following

$$\int_0^\infty \frac{1}{z^2+1}\,dz = \frac{\pi}{2}$$

proof

Put $z = \sqrt{t}$

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{t+1}\,dt$$

We can use the second integral representation by finding the values of x and y

$$x - 1 = \frac{-1}{2} \ \Rightarrow \ x = \frac{1}{2}$$

$$x + y = 1 \ \Rightarrow \ y = \frac{1}{2}$$

Hence we have

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)}\,dt = \frac{B(\frac{1}{2},\frac{1}{2})}{2} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{2} = \frac{\sqrt{\pi}\cdot\sqrt{\pi}}{2} = \frac{\pi}{2}$$

4.3 Example

$$\int_0^\infty \frac{1}{(z^2+1)^2}\,dz$$

Using the same substitution as the previous example we get

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^2}\,dt$$

Then we can find the values of x and y

$$x - 1 = \frac{-1}{2} \ \Rightarrow \ x = \frac{1}{2}$$

$$x + y = 2 \ \Rightarrow \ y = \frac{3}{2}$$

Then

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^2}\,dt = \frac{B(\frac{1}{2},\frac{3}{2})}{2} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})}{2} = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{4} = \frac{\pi}{4}$$

4.4 Example

Find the generalization

$$\int_0^\infty \frac{1}{(x^2+1)^n}\,dx\,,\ \forall\ n > \frac{1}{2}$$

Using the same substitution again

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^n}\,dt$$

Then we can find the values of x and y

$$x - 1 = \frac{-1}{2} \;\Rightarrow\; x = \frac{1}{2}$$

$$x + y = n \;\Rightarrow\; y = n - \frac{1}{2}$$

Then

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^n}\,dt = \frac{\Gamma(\frac{1}{2})\cdot\Gamma\left(n-\frac{1}{2}\right)}{2\Gamma(n)}$$

Now by LDF

$$\Gamma\left(n - \frac{1}{2}\right) = \frac{(2n-2)!\sqrt{\pi}}{4^{n-1}\,(n-1)!} = \frac{\Gamma\,(2n-1)\,\sqrt{\pi}}{4^{n-1}\,\Gamma(n)}$$

Substituting in our integral we have the following

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{-1}{2}}}{(t+1)^n}\,dt = \frac{2\pi\,\cdot\Gamma\,(2n-1)}{4^n\cdot\Gamma^2(n)}$$

$$\int_0^\infty \frac{1}{(x^2+1)^n}\,dx = \frac{\pi\,\cdot\Gamma\,(2n-1)}{2^{2n-1}\cdot\Gamma^2(n)}$$

It is easy to see that for $n \in \mathbb{Z}^+$ we get a π multiplied by some rational number.

4.5 Example

$$\int_0^1 \frac{z^n}{\sqrt{1-z}}\,dz = 2\cdot\,\frac{(2n)!!}{(2n+1)!!}$$

Where the double factorial !! is defined as the following

$$n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1 & ;\text{if } n \text{ is odd} \\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2 & ;\text{if } n \text{ is even} \\ 1 & ;\text{if } n = 0 \end{cases}$$

The integral in hand can be rewritten as

$$\int_0^1 z^n\cdot(1-z)^{-\frac{1}{2}}\,dz$$

We find the variables x and y

$$x - 1 = n \;\Rightarrow\; x = n + 1$$

$$y - 1 = \frac{-1}{2} \;\Rightarrow\; y = \frac{1}{2}$$

This can be written as

$$\int_0^1 z^n \cdot (1 - z)^{-\frac{1}{2}}\, dz = B(n + 1, \frac{1}{2})$$

By some simplifications

$$B\left(n + 1, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(n + 1)}{\Gamma\left(n + \frac{3}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma(n + 1)}{(n + \frac{1}{2})\Gamma\left(n + \frac{1}{2}\right)}$$

Now you shall realize that we must use LDF

$$\frac{\sqrt{\pi}\Gamma(n + 1)}{(n + \frac{1}{2})\Gamma\left(n + \frac{1}{2}\right)} = \frac{2\sqrt{\pi}\,n!}{(2n + 1)\sqrt{\pi}\,\frac{(2n)!}{4^n\,n!}} = \frac{2 \cdot 2^{2n}(n!)^2}{(2n)!}$$

Now we should separate odd and even terms in the denominator

$$2 \cdot \frac{2^{2n}(n \cdot (n - 1) \cdots 3 \cdot 2 \cdot 1)^2}{(2n \cdot (2n - 2)\, \cdots\, 4 \cdot 2)((2n + 1) \cdot (2n - 1) \cdots 3 \cdot 1)}$$

We insert 2^{2n} into the square to obtain

$$2 \cdot \frac{(2n \cdot (2n - 2) \cdots 6 \cdot 4 \cdot 2)^2}{(2n \cdot (2n - 2) \cdots 4 \cdot 2)((2n + 1) \cdot (2n - 1) \cdots 3 \cdot 1)} = 2 \cdot \frac{(2n)!!}{(2n + 1)!!}$$

4.6 Example

Find the following integral

$$\int_{-\infty}^{\infty} \left(1 + \frac{x^2}{n - 1}\right)^{-\frac{n}{2}}\, dx$$

First we shall realize the evenness of the integral

$$2\int_0^{\infty} \left(1 + \frac{x^2}{n - 1}\right)^{-\frac{n}{2}}\, dx$$

Let $t = \frac{x^2}{n-1}$

$$\sqrt{n - 1}\int_0^{\infty} \frac{t^{-\frac{1}{2}}}{(1 + t)^{\frac{n}{2}}}\, dt$$

Now we see that our integral becomes so familiar

$$\sqrt{n-1}\,B\left(\frac{1}{2},\frac{n-1}{2}\right)=\frac{\sqrt{\pi(n-1)}\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

4.7 Example

Find the following integral

$$\int_0^\infty \frac{x^{-p}}{x^3+1}\,dx$$

Let us do the substitution $x^3 = t$

$$\frac{1}{3}\int_0^\infty \frac{t^{-\frac{p+2}{3}}}{t+1}\,dt$$

Now we should find x, y

$$x=\frac{1-p}{3}$$

$$y+x=1 \;\Rightarrow\; y=1-\frac{1-p}{3}$$

so we have our beta representation of the integral

$$\frac{B\left(\frac{1-p}{3},\frac{1-p}{3}\right)}{3}=\frac{\Gamma\left(\frac{1-p}{3}\right)\Gamma\left(1-\frac{1-p}{3}\right)}{3}$$

Now we should use ERF

$$\frac{\Gamma\left(\frac{1-p}{3}\right)\Gamma\left(1-\frac{1-p}{3}\right)}{3}=\frac{\pi}{3\sin\left(\frac{\pi(1-p)}{3}\right)}=\frac{\pi}{3}\csc\left(\frac{\pi-\pi p}{3}\right)$$

4.8 Example

Now let us try to find

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin^3 z}\,dz$$

Rewrite as

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} z\cos^0 z\,dx$$

This is the Geometric representation

$$2x - 1 = \frac{3}{2} \;\Rightarrow\; x = \frac{5}{4}$$

$$2y - 1 = 0 \;\Rightarrow\; y = \frac{1}{2}$$

Then

$$\int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}} z\, dz = \frac{B\left(\frac{5}{4}, \frac{1}{2}\right)}{2} = \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{4}\right)}$$

4.9 Example

Find the following integral

$$\int_0^{\frac{\pi}{2}} (\sin z)^i \cdot (\cos z)^{-i}\, dz$$

This is the geometric representation

$$2x - 1 = i \;\Rightarrow\; x = \frac{1+i}{2}$$

$$2y - 1 = -i \;\Rightarrow\; y = \frac{1-i}{2}$$

Then

$$\frac{1}{2}\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(\frac{1-i}{2}\right)$$

Now we see that we have to use ERF

$$\frac{1}{2}\Gamma\left(\frac{1+i}{2}\right)\Gamma\left(1 - \frac{1+i}{2}\right) = \frac{\pi}{2\sin\left(\frac{\pi(1+i)}{2}\right)} = \frac{\pi}{2}\text{sech}\,\left(\frac{\pi}{2}\right)$$

4.10 Exercise

Prove

$$\int_0^{\infty} \frac{x^{2m+1}}{(ax^2 + c)^n} dx = \frac{m!\,(n-m-2)!}{2(n-1)!\,a^{m+1}\,c^{n-m-1}}$$

5 Digamma function

5.1 Definition

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

We call digamma function the logarithmic derivative of the gamma function. Using this we can define the derivative of the gamma function.

$$\Gamma'(x) = \psi(x)\,\Gamma(x)$$

5.2 Example

Find the derivative of

$$f(x) = \frac{\Gamma(2x+1)}{\Gamma(x)}$$

We can use the differentiation rule for quotients

$$\frac{2\Gamma'(2x+1)\Gamma(x) - \Gamma'(x)\Gamma(2x+1)}{\Gamma^2(x)}$$

which can be rewritten as

$$\frac{2\Gamma(2x+1)\psi(2x+1)\Gamma(x) - \psi(x)\Gamma(x)\Gamma(2x+1)}{\Gamma^2(x)} = \frac{\Gamma(2x+1)}{\Gamma(x)}\left(2\psi(2x+1) - \psi(x)\right)$$

5.3 Difference formulas

5.3.1 First difference formula

$$\psi(1-x) - \psi(x) = \pi\cot(\pi x)$$

proof

We know by ERF that

$$\Gamma(x)\Gamma(1-x) = \pi\csc(\pi x)$$

Now differentiate both sides

$$\psi(x)\Gamma(x)\Gamma(1-x) - \psi(1-x)\Gamma(x)\Gamma(1-x) = -\pi^2 \csc(\pi x)\, \cot(\pi x)$$

Which can be simplified

$$\Gamma(x)\Gamma(1-x)\left(\psi(1-x) - \psi(x)\right) = \pi^2 \csc(\pi x)\, \cot(\pi x)$$

Further simplifications using ERF results in

$$\psi(1-x) - \psi(x) = \pi \cot(\pi x)$$

5.3.2 Second difference formula

$$\psi(1+x) - \psi(x) = \frac{1}{x}$$

proof

Let us start by the following

$$\frac{\Gamma(1+x)}{\Gamma(x)} = x$$

Now differentiate both sides

$$\frac{\Gamma(1+x)}{\Gamma(x)}\left(\psi(1+x) - \psi(x)\right) = 1$$

Which simplifies to

$$\psi(1+x) - \psi(x) = \frac{\Gamma(x)}{\Gamma(1+x)} = \frac{1}{x}$$

5.4 Example

Find the following integral

$$\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\, dx$$

Consider the general case

$$\int_0^\infty \frac{x^a}{(1+x^2)^2}\, dx$$

Use the following substitution $x^2 = t$

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{a-1}{2}}}{(1+t)^2}\,dt$$

By the beta function this is equivalent to

$$\frac{1}{2}\int_0^\infty \frac{t^{\frac{a-1}{2}}}{(1+t)^2}\,dt = \frac{1}{2}B\left(\frac{a+1}{2}, 2-\frac{a+1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(2-\frac{a+1}{2}\right)$$

Differentiate with respect to a

$$F'(a) = \frac{1}{4}\int_0^\infty \frac{\log(t)\,t^{\frac{a-1}{2}}}{(1+t)^2}\,dt = \frac{1}{4}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(2-\frac{a+1}{2}\right)\left[\psi\left(\frac{a+1}{2}\right)-\psi\left(2-\frac{a+1}{2}\right)\right]$$

Now put $a = 0$

$$\frac{1}{4}\int_0^\infty \frac{\log(t)\,t^{\frac{-1}{2}}}{(1+t)^2}\,dt = \frac{1}{4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)\left[\psi\left(\frac{1}{2}\right)-\psi\left(\frac{3}{2}\right)\right]$$

Now we use our second difference formula

$$\psi\left(\frac{1}{2}\right)-\psi\left(\frac{3}{2}\right) = -\left(\psi\left(1+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\right)\right) = -2$$

Also by some gamma manipulation we have

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

The integral reduces to

$$\frac{1}{4}\int_0^\infty \frac{\log(t)\,t^{\frac{-1}{2}}}{(1+t)^2}\,dt = -\frac{\pi}{4}$$

Putting $x^2 = t$ we have our result

$$\int_0^\infty \frac{\log(x)}{(1+x^2)^2}\,dx = -\frac{\pi}{4}$$

5.5 Series Representation

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^\infty \frac{x}{n(n+x)}$$

proof

We start by taking the logarithm of the Weierstrass representation of the gamma function

$$\log\left(\Gamma(x)\right) = -\gamma\,x - \log(x) + \sum_{n=1}^\infty -\log\left(1+\frac{x}{n}\right)+\frac{x}{n}$$

Now we shall differentiate with respect to x

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{\frac{-1}{n}}{1+\frac{x}{n}} + \frac{1}{n}$$

Further simplification will result in the following

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}$$

5.6 Some Values

Find the values of

1. $\psi(1)$

$$\psi(1) = -\gamma - 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

It should be easy to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Hence we have

$$\psi(1) = -\gamma$$

2. $\psi\left(\frac{1}{2}\right)$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2 + \sum_{n=1}^{\infty} \frac{1}{n(2n+1)}$$

We need to find

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)}$$

We can start by

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$$

So we can prove easily that

$$\sum_{n=1}^{\infty} \frac{1}{n(2n+1)} = 2 - 2\log(2)$$

Hence

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\log(2)$$

5.7 Example

Prove that

$$\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\, dx = \gamma$$

proof

Let $x = e^{-t}$

$$\int_0^{\infty} \frac{1}{e^t - 1} - \frac{e^{-t}}{t}\, dt$$

Let the following

$$F(s) = \int_0^{\infty} \frac{t^s}{e^t - 1} - t^{s-1}e^{-t}\, dt = \zeta(s+1)\Gamma(s+1) - \Gamma(s)$$

Hence the limit

$$\lim_{s\to 0} \Gamma(s+1)\left(\zeta(s+1) - \frac{1}{s}\right) = \lim_{s\to 0} \zeta(s+1) - \frac{1}{s}$$

Use the expansion of the zeta function

$$\zeta(s+1) = \frac{1}{s} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!}(-s)^n$$

Hence the limit is equal to $\gamma_0 = \gamma$.

5.8 Integral representations

5.8.1 First Integral representation

$$\psi(a) = \int_0^{\infty} \frac{e^{-z} - (1+z)^{-a}}{z}\, dz$$

We begin with the double integral

$$\int_0^\infty \int_1^t e^{-xz}\, dx\, dz = \int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\, dz$$

Using fubini theorem we also have

$$\int_1^t \int_0^\infty e^{-xz}\, dz\, dx = \int_1^t \frac{1}{x}\, dx = \log t$$

Hence we have the following

$$\int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\, dz = \log(t)$$

We also know that

$$\Gamma'(a) = \int_0^\infty t^{a-1}e^{-t}\, \log t\, dt$$

Hence we have

$$\Gamma'(a) = \int_0^\infty t^{a-1}e^{-t} \left(\int_0^\infty \frac{e^{-z} - e^{-tz}}{z}\, dz\right) dt = \int_0^\infty \int_0^\infty \frac{t^{a-1}e^{-t}e^{-z} - t^{a-1}e^{-t(z+1)}}{z}\, dz\, dt$$

Now we can use the fubini theorem

$$\Gamma'(a) = \int_0^\infty \int_0^\infty \frac{t^{a-1}e^{-t}e^{-z} - t^{a-1}e^{-t(z+1)}}{z}\, dt\, dz$$

$$\Gamma'(a) = \int_0^\infty \frac{1}{z}\left(e^{-z}\int_0^\infty t^{a-1}e^{-t}\, dt - \int_0^\infty t^{a-1}e^{-t(z+1)}\, dt\right) dz$$

But we can easily deduce using Laplace that

$$\int_0^\infty t^{a-1}e^{-t(z+1)}\, dt = \Gamma(a)\,(z+1)^{-a}$$

Aslo we have

$$\int_0^\infty t^{a-1}e^{-t}\, dt = \Gamma(a)$$

Hence we can simplify our integral to the following

$$\Gamma'(a) = \Gamma(a)\int_0^\infty \frac{e^{-z} - (1+z)^{-a}}{z} dz$$

$$\frac{\Gamma'(a)}{\Gamma(a)} = \psi(a) = \int_0^\infty \frac{e^{-z} - (1+z)^{-a}}{z}\, dz$$

5.8.2 Second Integral representation

$$\psi(s+1) = -\gamma + \int_0^1 \frac{1-x^s}{1-x}\,dx$$

proof

This can be done by noting that

$$\psi(s+1) = -\gamma + \sum_{n=1}^{\infty} \frac{s}{n(n+s)}$$

It is left as an exercise to prove that

$$\sum_{n=1}^{\infty} \frac{s}{n(n+s)} = \int_0^1 \frac{1-x^s}{1-x}\,dx$$

5.8.3 Third Integral representation

$$\psi(a) = \int_0^{\infty} \frac{e^{-t}}{t} - \frac{e^{-(at)}}{1-e^{-t}}\,dt$$

proof

Let $e^{-t} = x$

$$\int_0^1 -\frac{1}{\log(x)} - \frac{x^{a-1}}{1-x}\,dx$$

By adding and subtracting 1

$$-\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\,dx + \int_0^1 \frac{1-x^{a-1}}{1-x}\,dx$$

Using the second integral representation

$$-\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\,dx + \gamma + \psi(a)$$

We have already proved that

$$\int_0^1 \frac{1}{\log(x)} + \frac{1}{1-x}\,dx = \gamma$$

Finally we get

$$\int_0^{\infty} \frac{e^{-t}}{t} - \frac{e^{-(at)}}{1-e^{-t}}\,dt = -\gamma + \gamma + \psi(a) = \psi(a)$$

5.8.4 Fourth Integral representation

Prove that

$$\psi(z) = \log(z) - \frac{1}{2z} - 2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi}-1)}dt \;\; ; \; \text{Re}z > 0$$

We prove that

$$2\int_0^\infty \frac{t}{(t^2+z^2)(e^{2\pi}-1)}dt = \log(z) - \frac{1}{2z} - \psi(z)$$

First note that

$$\frac{2}{e^{2\pi t}-1} = \coth(\pi t) - 1$$

Also note that

$$\coth(\pi t) = \frac{1}{\pi t} + \frac{2t}{\pi}\sum_{k=1}^\infty \frac{1}{k^2+t^2}$$

Hence we conclude that

$$\frac{2t}{e^{2\pi t}-1} = \frac{1}{\pi} - t + \frac{2t^2}{\pi}\sum_{k=1}^\infty \frac{1}{k^2+t^2}$$

Substitute the value in the integral

$$\int_0^\infty \frac{1}{t^2+z^2}\left\{\frac{1}{\pi} - t + \frac{2t^2}{\pi}\sum_{k=1}^\infty \frac{1}{k^2+t^2}\right\}dt$$

The first integral

$$\frac{1}{\pi}\int_0^\infty \frac{1}{t^2+z^2}dt = \frac{1}{2z}$$

Since the second integral is divergent we put

$$\int_0^N \frac{t}{t^2+z^2}\,dt = \frac{1}{2}\log(N^2+z^2) - \log(z)$$

Also for the series

$$\frac{2}{\pi}\sum_{k=1}^N \int_0^\infty \frac{t^2}{(t^2+z^2)(t^2+k^2)}dt = \sum_{k=1}^N \frac{1}{k+z}$$

Which simplifies to

$$\sum_{k=1}^{N}\frac{1}{k+z}=\sum_{k=1}^{N}\frac{1}{k}-\sum_{k=1}^{N}\frac{z}{k(k+z)}=H_N-\sum_{k=1}^{N}\frac{z}{k(k+z)}$$

Now take the limit

$$\lim_{N\to\infty}-\frac{1}{2}\log(N^2+z^2)+\log(z)+H_N-\sum_{k=1}^{N}\frac{z}{k(k+z)}$$

Or

$$\lim_{N\to\infty}H_N-\log(N)+\log(z)-\sum_{k=1}^{\infty}\frac{z}{k(k+z)}$$

This simplifies to

$$\log(z)+\gamma-\sum_{k=1}^{\infty}\frac{z}{k(k+z)}=\log(z)-\frac{1}{z}-\psi(z)$$

Collecting the results we have

$$2\int_0^{\infty}\frac{t}{(t^2+z^2)(e^{2\pi}-1)}dt=\log(z)+\frac{1}{2z}-\frac{1}{z}-\psi(z)=\log(z)-\frac{1}{2z}-\psi(z)$$

5.9 Gauss Digamma theorem

Let p/q be a rational number with $0<p<q$ then

$$\psi\left(\frac{p}{q}\right)=-\gamma-\log(2q)-\frac{\pi}{2}\cot\left(\frac{p}{q}\pi\right)+2\sum_{k=1}^{q/2-1}\cos\left(\frac{2\pi pk}{q}\right)\log\left[\sin\left(\frac{\pi k}{q}\right)\right]$$

proof

The proof is omitted.

5.10 More results

Assume that $p=1$ and $q>1$ is an integer then

$$\psi\left(\frac{1}{q}\right)=-\gamma-\log(2q)-\frac{\pi}{2}\cot\left(\frac{\pi}{q}\right)+2\sum_{k=1}^{q/2-1}\cos\left(\frac{2\pi k}{q}\right)\log\left[\sin\left(\frac{\pi k}{q}\right)\right]$$

So for example

$$\psi\left(\frac{1}{3}\right) = \frac{1}{6}(-6\gamma - \pi\sqrt{3} - 9\log(3))$$
$$\psi\left(\frac{1}{4}\right) = \frac{1}{2}(-2\gamma - \pi - 6\log(2))$$
$$\psi\left(\frac{1}{6}\right) = -\gamma - \frac{1}{2}\sqrt{3}\pi - 2\log(2) - \frac{3}{2}\log(3)$$

5.11 Example

Find the following integral

$$\int_0^\infty e^{-at}\,\log(t)\,dt$$

We start by considering

$$F(b) = \int_0^\infty e^{-at}\,t^b\,dt$$

Now use the substitution $x = at$ we get

$$F(b) = \frac{1}{a}\int_0^\infty e^{-x}\left(\frac{x}{a}\right)^b\,dx$$

We can use the gamma function

$$F(b) = \frac{1}{a}\int_0^\infty e^{-x}\left(\frac{x}{a}\right)^b\,dx = \frac{\Gamma(b+1)}{a^{b+1}}$$

Now differentiate with respect to b

$$F'(b) = \frac{1}{a}\int_0^\infty e^{-x}\,\log\left(\frac{x}{a}\right)\left(\frac{x}{a}\right)^b\,dx = \frac{\Gamma(b+1)\psi(b+1)}{a^{b+1}} - \frac{\log(a)\Gamma(b+1)}{a^{b+1}}$$

Now put $b = 0$ and $at = x$

$$\int_0^\infty e^{-at}\,\log(t)\,dt = \frac{\psi(1) - \log(a)}{a} = -\frac{\gamma + \log(a)}{a}$$

5.12 Example

Prove the following

$$\int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\log x)}dx = \log\left\{\frac{\Gamma(b+c+1)\Gamma(c+a+1)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(a+b+c+1)}\right\}$$

proof

First note that since there is a log in the denominator that gives as an idea to use differentiation under the integral sign.

Let

$$F(c) = \int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\log x)}dx$$

Differentiate with respect to c

$$F'(c) = \int_0^1 \frac{(1-x^a)(1-x^b)x^c}{(1-x)}dx$$

By expanding

$$F'(c) = \int_0^1 \frac{(1-x^a-x^b+x^{a+b})x^c}{(1-x)}dx = \int_0^1 \frac{x^c - x^{a+c} - x^{b+c} + x^{a+b+c}}{(1-x)}dx$$

We can add and subtract one to use the second integral representation

$$F'(c) = \int_0^1 \frac{(x^c - 1) + (1 - x^{a+c}) + (1 - x^{b+c}) + (x^{a+b+c} - 1)}{(1-x)}dx$$

Distribute the integral over the terms

$$F'(c) = -\int_0^1 \frac{1-x^c}{1-x}\,dx + \int_0^1 \frac{1-x^{a+c}}{1-x}\,dx + \int_0^1 \frac{1-x^{b+c}}{1-x}\,dx - \int_0^1 \frac{1-x^{a+b+c}}{1-x}dx$$

Which simplifies to

$$F'(c) = -\psi(c+1) + \psi(a+c+1) + \psi(b+c+1) - \psi(a+b+c+1)$$

Integrate with respect to c

$$F(c) = -\log\left[\Gamma(c+1)\right] + \log\left[\Gamma(a+c+1)\right] + \log\left[\Gamma(b+c+1)\right] - \log\left[\Gamma(a+b+c+1)\right] + e$$

Which reduces to

$$\log\left[\frac{\Gamma(a+c+1)\Gamma(b+c+1)}{\Gamma(c+1)\Gamma(a+b+c+1)}\right] + e$$

Now put $c = 0$ we have

$$0 = \log\left[\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\right] + e$$

The constant

$$e = -\log\left[\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\right]$$

So we have the following

$$F(c) = \log\left[\frac{\Gamma(a+c+1)\Gamma(b+c+1)}{\Gamma(c+1)\Gamma(a+b+c+1)}\right] - \log\left[\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}\right]$$

Hence we have the result

$$\int_0^1 \frac{(1-x^a)(1-x^b)(1-x^c)}{(1-x)(-\log x)}dx = \log\left\{\frac{\Gamma(b+c+1)\Gamma(c+a+1)\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)\Gamma(c+1)\Gamma(a+b+c+1)}\right\}$$

5.13 Example

Find the following integral

$$\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax}\right)\frac{dx}{x}$$

Let us first use the substitution $t = ax$

$$\int_0^\infty \left(e^{-\frac{bt}{a}} - \frac{1}{1+t}\right)\frac{dt}{t}$$

Add and subtract e^{-t}

$$\int_0^\infty \left(e^{-t} - e^{-t} + e^{-\frac{bt}{a}} - \frac{1}{1+t}\right)\frac{dt}{t}$$

Separate into two integrals

$$\int_0^\infty \left(e^{-t} - \frac{1}{1+t}\right)\frac{dt}{t} + \int_0^\infty \frac{e^{-\frac{bt}{a}} - e^{-t}}{t}\,dt$$

The first integral is a representation of the Euler constant when $a = 1$

$$\int_0^\infty \left(e^{-t} - \frac{1}{1+t}\right)\frac{dt}{t} = -\gamma$$

We also proved

$$\int_0^\infty \frac{e^{-\frac{bt}{a}} - e^{-t}}{t}\,dt = -\log\left(\frac{b}{a}\right) = \log\left(\frac{a}{b}\right)$$

Hence the result

$$\int_0^\infty \left(e^{-bx} - \frac{1}{1+ax}\right)\frac{dx}{x} = \log\left(\frac{a}{b}\right) - \gamma$$

5.14 Example

Find the following integral

$$\int_0^\infty e^{-ax}\left(\frac{1}{x} - \coth(x)\right)\,dx$$

By using the exponential representation of the hyperbolic functions

$$\int_0^\infty e^{-ax}\left(\frac{1}{x} - \frac{1+e^{-2x}}{1-e^{-2x}}\right)\,dx$$

Now let $2x = t$ so we have

$$\int_0^\infty e^{-\left(\frac{at}{2}\right)}\left(\frac{1}{t} - \frac{1+e^{-t}}{2(1-e^{-t})}\right)\,dt$$

$$\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}}{t} - \frac{e^{-\left(\frac{at}{2}\right)} + e^{\left(-\frac{at}{2}-t\right)}}{2(1-e^{-t})}\,dt$$

By adding and subtracting some terms

$$\int_0^\infty \frac{e^{-t} + e^{-\left(\frac{at}{2}\right)} - e^{-t}}{t} - \frac{e^{-\left(\frac{at}{2}\right)} + e^{-\left(\frac{at}{2}\right)} - e^{-\frac{at}{2}} + e^{-\left(\frac{at}{2}\right)-t}}{2(1-e^{-t})}\,dt$$

Separate the integrals

$$\int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-\left(\frac{at}{2}\right)}}{1-e^{-t}}\,dt + \int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)} - e^{\left(-\frac{at}{2}-t\right)}}{2(1-e^{-t})}\,dt + \int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)} - e^{-t}}{t}\,dt$$

By using the third integral representation

$$\int_0^\infty \frac{e^{-t}}{t} - \frac{e^{-\left(\frac{at}{2}\right)}}{1-e^{-t}}\,dt = \psi\left(\frac{a}{2}\right)$$

The second integral reduces to

$$\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)} - e^{-\left(\frac{at}{2}-t\right)}}{2(1-e^{-t})} = \int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)}}{2}\,dt = \frac{1}{a}$$

The third integral

$$\int_0^\infty \frac{e^{-\left(\frac{at}{2}\right)} - e^{-t}}{t}\,dt = -\log\left(\frac{a}{2}\right)$$

By collecting the results

$$\int_0^\infty e^{-ax}\left(\frac{1}{x} - \coth(x)\right)\,dx = \psi\left(\frac{a}{2}\right) - \log\left(\frac{a}{2}\right) + \frac{1}{a}$$

5.15 Example

Prove that

$$\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(ax)} = \frac{1}{2}\psi\left(\frac{1}{2}+\frac{a}{2\pi}\right) - \frac{1}{2}\psi\left(\frac{a}{2\pi}\right) - \frac{2}{\pi}a$$

proof

$$\begin{aligned}
\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(ax)} &= \int_0^\infty \frac{x}{x^2+a^2}\frac{dx}{\sinh(x)} \\
&= \int_0^\infty \int_0^\infty e^{-at}\frac{\sin(xt)}{\sinh(x)}\,dt\,dx \\
&= \int_0^\infty e^{-at}\int_0^\infty \frac{\sin(xt)}{\sinh(x)}\,dx\,dt \\
&= \frac{\pi}{2}\int_0^\infty e^{-at}\tanh\left(\frac{\pi}{2}t\right)dt \\
&= \int_0^\infty e^{-zx}\tanh(x)\,dt \quad ; z = \frac{2}{\pi}a \\
&= \int_0^\infty \frac{e^{-zx}(1-e^{-2x})}{e^{-2x}+1}\,dx
\end{aligned}$$

By splitting the integral we have

$$\begin{aligned}
\int_0^\infty \frac{e^{-zx}}{e^{-2x}+1}\,dx &= \sum_{n\geq 0}\int_0^\infty e^{-x(2n+z)}\,dx \\
&= \sum_{n\geq 0}\frac{(-1)^n}{2n+z} \\
&= \frac{1}{4}\left(\psi\left(\frac{1}{2}+\frac{z}{4}\right) - \psi\left(\frac{z}{4}\right)\right)
\end{aligned}$$

$$\begin{aligned}
-\int_0^\infty \frac{-e^{-x(z+2)}}{e^{-2x}+1}\,dx &= -\sum_{n\geq 0}\frac{(-1)^n}{z+2+2n} \\
&= -\frac{1}{4}\left(-\psi\left(\frac{1}{2}+\frac{z}{4}\right) + \psi\left(1+\frac{z}{4}\right)\right)
\end{aligned}$$

Hence we have

$$\begin{aligned}
\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(ax)} &= \frac{1}{4}\left(2\psi\left(\frac{1}{2}+\frac{z}{4}\right) - \psi\left(1+\frac{z}{4}\right) - \psi\left(\frac{z}{4}\right)\right) \\
&= \frac{1}{2}\psi\left(\frac{1}{2}+\frac{z}{4}\right) - \frac{1}{2}\psi\left(\frac{z}{4}\right) - z \\
&= \frac{1}{2}\psi\left(\frac{1}{2}+\frac{a}{2\pi}\right) - \frac{1}{2}\psi\left(\frac{a}{2\pi}\right) - \frac{2}{\pi}a
\end{aligned}$$

Let $a = \pi/2$

$$\int_0^\infty \frac{x}{x^2+1}\frac{dx}{\sinh(\frac{\pi}{2}x)} = \frac{1}{2}\psi\left(\frac{1}{2}+\frac{1}{4}\right) - \frac{1}{2}\psi\left(\frac{1}{4}\right) - 1$$
$$= \frac{\pi}{2}\cot(\pi/4) - 1$$
$$= \frac{\pi}{2} - 1$$

6 Zeta function

Zeta function is one of the most important mathematical functions. The study of zeta function isn't exclusive to analysis. It also extends to number theory and the most celebrating theorem of Riemann.

6.1 Definition

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

6.2 Bernoulli numbers

We define the Bernoulli numbers B_k as

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

Now let us derive some values for the Bernoulli numbers , rewrite the power series as

$$x = (e^x - 1) \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

By expansion

$$x = \left(x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \right) \cdot \left(B_0 + B_1\, x + \frac{B_2}{2!}\, x^2 + \frac{B_3}{3!} x^3 + \cdots \right)$$

By multiplying we get

$$x = B_0 x + \left(B_1 + \frac{B_0}{2!} \right) x^2 + \left(\frac{B_0}{3!} + \frac{B_2}{2!} + \frac{B_1}{2!} \right) x^3 + \left(\frac{B_0}{4!} + \frac{B_1}{3!} + \frac{B_2}{2!\,2!} + \frac{B_3}{3!} \right) x^4 + \cdots$$

By comparing the terms we get the following values

$$B_0 = 1\,,\; B_1 = -\frac{1}{2} \;+\; B_2 = \frac{1}{6}\,,\; B_3 = 0\,,\; B_4 = -\frac{1}{30}, \cdots$$

Actually we also deduce that

$$B_{2k+1} = 0 \quad , \;\forall \quad k \in \mathbb{Z}^+$$

6.3 Relation between zeta and Bernoulli numbers

According to Euler we have the following relation

$$\zeta(2k) = (-1)^{k-1} B_{2k} \frac{2^{2k-1}}{(2k)!} \pi^{2k}$$

proof

We start by the product formula of the sine function

$$\frac{\sin(z)}{z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

Take the logarithm to both sides

$$\log(\sin(z)) - \log(z) = \sum_{n=1}^{\infty} \log\left(1 - \frac{z^2}{n^2 \pi^2}\right)$$

By differentiation with respect to z

$$\cot(z) - \frac{1}{z} = -2 \sum_{n=1}^{\infty} \frac{\frac{z}{n^2 \pi^2}}{1 - \frac{z^2}{n^2 \pi^2}}$$

By simple algebraical manipulation we have

$$z \cot(z) = 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2 \pi^2} \left(\frac{1}{1 - \frac{z^2}{\pi^2 n^2}} \right)$$

Now using the power series expansion

$$\frac{1}{1 - \frac{z^2}{\pi^2 n^2}} = \sum_{k=0}^{\infty} \frac{1}{n^{2k} \pi^{2k}} z^{2k} \quad , |z| < \pi n$$

$$\frac{z^2}{n^2 \pi^2} \left(\frac{1}{1 - \frac{z^2}{\pi^2 n^2}} \right) = \sum_{k=0}^{\infty} \frac{1}{n^{2k+2} \pi^{2k+2}} z^{2k+2} = \sum_{k=1}^{\infty} \frac{1}{n^{2k} \pi^{2k}} z^{2k}$$

So the sums becomes

$$z \cot(z) = 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^{2k}} \frac{z^{2k}}{\pi^{2k}}$$

Now if we invert the order of summation we have

$$z \cot(z) = 1 - 2 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \frac{z^{2k}}{\pi^{2k}} = 1 - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k}$$

Euler didn't stop here, he used power series for $z \cot(z)$ using the Bernoulli numbers.

Start by the equation

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$

By putting $x = 2iz$ we have

$$\frac{2iz}{e^{2iz} - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} (2iz)^k$$

Which can be reduced directly to the following by noticing that $B_{2k+1} = 0$

$$z \cot(z) = 1 - \sum_{k=1}^{\infty} (-1)^{k-1} B_{2k} \frac{2^{2k}}{(2k)!} z^{2k}$$

The result is immediate by comparing the two different representations.

6.4 Exercise

Find the values of

$$\zeta(4) \, , \, \zeta(6) \, , \, B_5 \, , \, B_6$$

6.5 Integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt$$

proof

Start by the integral representation

$$\int_0^{\infty} \frac{e^{-t} t^{s-1}}{1 - e^{-t}} \, dt$$

Using the power expansion

$$\frac{1}{1 - e^{-t}} = \sum_{n=0}^{\infty} e^{-nt}$$

Hence we have

$$\int_0^{\infty} e^{-t} t^{s-1} \left(\sum_{n=0}^{\infty} e^{-nt} \right) dt$$

By swapping the series and integral

$$\sum_{n=0}^{\infty} \int_0^{\infty} t^{s-1} e^{-(n+1)t} \, dt = \Gamma(s) \sum_{n=0}^{\infty} \frac{1}{(n+1)^s} = \Gamma(s)\zeta(s)$$

6.6 Hurwitz zeta and polygamma functions

Hurwitz zeta is a generalization of the zeta function by adding a parameter .

6.6.1 Definition

$$\zeta(a,z) \; = \; \sum_{n=0}^{\infty} \frac{1}{(n+z)^a} \;;\zeta(a,1) = \zeta(a)$$

Let us define the polygamma function as the function produced by differentiating the Digamma function and it is often denoted by

$$\psi_n(z) \;\; \forall \; n \geq 0$$

We define the digamma function by setting $n = 0$ so it's denoted by $\psi_0(z)$.

Other values can be found by the following recurrence relation

$$\psi'_n(z) = \psi_{n+1}(z)$$

So we have

$$\psi_1(z) = \psi'_0(z)$$

6.6.2 Relation between zeta and polygamma

$\forall \; n \geq 1$

$$\psi_n(z) \; = \; (-1)^{n+1} n! \, \zeta(n+1, z)$$

$$\psi_{2n-1}(1) = (-1)^{n-1} B_{2n} \frac{2^{2n-2}}{n} \pi^{2n}$$

proof

We have already proved the following relation

$$\psi_0(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$$

This can be written as the following

$$\psi_0(z) = -\gamma + \sum_{k=0}^{\infty} \frac{1}{k+1} - \frac{1}{k+z}$$

By differentiating with respect to z

$$\psi_1(z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^2}$$

$$\psi_2(z) = -2\sum_{k=0}^{\infty} \frac{1}{(k+z)^3}$$

$$\psi_3(z) = 2\cdot 3\sum_{k=0}^{\infty} \frac{1}{(k+z)^4}$$

$$\psi_4(z) = -2\cdot 3\cdot 4\sum_{k=0}^{\infty} \frac{1}{(k+z)^5}$$

Continue like that to obtain

$$\psi_n(z) = (-1)^{n+1} n!\sum_{k=0}^{\infty} \frac{1}{(k+z)^{n+1}}$$

We realize the RHS is just the Hurwitz zeta function

$$\psi_n(z) = (-1)^{n+1} n!\,\zeta(n+1, z)$$

By setting $z = 1$ we have an equation in terms of the ordinary zeta function

$$\psi_n(1) = (-1)^{n+1} n!\,\zeta(n+1)$$

Now since we already proved in the preceding section that

$$\zeta(2k) \ = \ (-1)^{k-1} B_{2k} \frac{2^{2k-1}}{(2k)!} \pi^{2k}$$

we can easily verify the following

$$\psi_{2n-1}(1) = (2n-1)!\,(-1)^{n-1} B_{2n} \frac{2^{2n-1}}{(2n)!} \pi^{2n} = (-1)^{n-1} B_{2n} \frac{2^{2n-2}}{n} \pi^{2n}$$

This can be used to deduce some values for the polygamma function

$$\psi_1(1) = \frac{\pi^2}{6}\ ,\ \psi_3(1) = \frac{\pi^4}{15}$$

Other values can be evaluated in terms of the zeta function

$$\psi_2(1) = -2\zeta(3)\ ,\ \psi_4(1) = -24\zeta(5)$$

6.7 Example

Prove that

$$\int_0^{\frac{\pi}{2}} x\sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx = \frac{\pi}{16} - \frac{\pi^3}{192}$$

proof

Start by the transformation $x \to \frac{\pi}{2} - x$

$$\int_0^{\frac{\pi}{2}} x\sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx = \frac{\pi}{4}\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx$$

We need to find

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx$$

Let us start by the following

$$F(a,b) = 2\int_0^{\frac{\pi}{2}} \sin^{2a-1}(x)\cos^{2b-1}(x)\,dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Now let us differentiate with respect to a

$$\frac{\partial}{\partial a}(F(a,b)) = 4\int_0^{\frac{\pi}{2}} \sin^{2a-1}(x)\cos^{2b-1}(x)\log(\sin x)\,dx = \frac{\Gamma(a)\Gamma(b)\left(\psi_0(a) - \psi_0(a+b)\right)}{\Gamma(a+b)}$$

Differentiate again but this time with respect to b

$$\frac{\partial}{\partial b}\left(F_a(a,b)\right) = 8\int_0^{\frac{\pi}{2}} \sin^{2a-1}(x)\cos^{2b-1}(x)\log(\sin x)\,\log(\cos x)dx$$

$$= \frac{\Gamma(a)\Gamma(b)\left(\psi_0^2(a+b) + \psi_0(a)\psi_0(b) - \psi_0(a)\psi_0(a+b) - \psi_0(b)\psi_0(a+b) + \psi_1(a+b)\right)}{\Gamma(a+b)}$$

Putting $a = b = 1$ we have the following

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\,\log(\cos x)dx = \frac{\psi_0^2(2) + \psi_0^2(1) - \psi_0(1)\psi_0(2) - \psi_0(1)\psi(2) - \psi_1(2)}{8}$$

By simple algebra we arrive to

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\,\log(\cos x)dx = \frac{(\psi_0(2) - \psi_0(1))^2 - \psi_1(2)}{8}$$

We already know that $\psi_0(1) = -\gamma$ and $\psi_0(2) = 1 - \gamma$

Now to evaluate $\psi_1(2)$, we have to use the zeta function we have already established the following relation

$$\psi_1(z) = \sum_{k=0}^{\infty} \frac{1}{(n+z)^2}$$

Now putting $z = 2$ we have the following

$$\psi_1(2) = \sum_{k=0}^{\infty} \frac{1}{(k+2)^2}$$

Let us write the first few terms in the expansion

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

we see this is similar to $\zeta(2)$ but we are missing the first term

$$\psi_1(2) = \zeta(2) - 1 = \frac{\pi^2}{6} - 1$$

Collecting all these results together we have

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\log(\sin x)\ \log(\cos x)dx\ = \frac{1}{4} - \frac{\pi^2}{48}$$

Finally we get our result

$$\int_0^{\frac{\pi}{2}} x\sin(x)\cos(x)\log(\sin x)\log(\cos x)\,dx = \frac{\pi}{16} - \frac{\pi^3}{192}$$

7 Dirichlet eta function

Dirichlet eta function is the alternating form of the zeta function.

7.1 Definition

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

The alternating form of the zeta function is easier to compute once we have established the main results of the zeta function because the alternating form is related to the zeta function through the relation

7.2 Relation to Zeta function

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s)$$

proof

We will start by the RHS

$$\left(1 - 2^{1-s}\right)\zeta(s) = \zeta(s) - 2^{1-s}\zeta(s)$$

Which can be written as sums of series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{2^{s-1}} \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

Clearly we can see that we are subtracting even terms twice , this is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

This looks easier to understand if we write the terms

$$\left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots\right) - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right)$$

Rearranging the terms we establish the alternating form

$$1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \eta(s)$$

7.3 Integral representation

$$\eta(s)\,\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t + 1}\,dt$$

proof

Start by the RHS

$$\int_0^\infty \frac{t^{s-1}}{e^t + 1}\,dt = \int_0^\infty \frac{e^{-t}t^{s-1}}{1 + e^{-t}}\,dt$$

Now using the power expansion we arrive to

$$\int_0^\infty e^{-t}t^{s-1}dt\left(\sum_{n=0}^{\infty}(-1)^n e^{-nt}\right)$$

$$\sum_{n=0}^{\infty}(-1)^n \int_0^\infty e^{-(n+1)t}t^{s-1}dt$$

Using Laplace transform we can solve the inner integral

$$\int_0^\infty e^{-(n+1)t}t^{s-1}dt = \frac{\Gamma(s)}{(n+1)^s}$$

Hence we have the following

$$\Gamma(s)\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^s} = \Gamma(s)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s} = \Gamma(s)\,\eta(s)$$

An easy result of the above integral

$$\int_0^\infty \frac{t}{e^t + 1}\,dt = \Gamma(2)\,\eta(2) = \frac{\pi^2}{12}$$

8 Polylogarithm

8.1 Definition

$$\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

The name contains two parts, (poly) because we can choose different n and produce many functions and (logarithm) because we can express $\mathrm{Li}_1(z) = -\log(1-z)$.

8.2 Relation to other functions

We can relate it to the Zeta function

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n)$$

In particular we have for $n = 2$

$$\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$$

Also we can relate it to the eta function though $z = -1$

$$\mathrm{Li}_n(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^n} = -\eta(n)$$

Also we can relate it to logarithms by putting $n = 1$

$$\mathrm{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^k}{k}$$

The power expansion on the left is famous

$$\sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1-z)$$

8.3 Integral representation

$$\mathrm{Li}_{n+1}(z) = \int_0^z \frac{\mathrm{Li}_n(t)}{t}\, dt$$

proof

Using the series representation we have

$$\int_0^z \frac{1}{t}\left(\sum_{k=1}^{\infty} \frac{t^k}{k^n}\right) dt = \sum_{k=1}^{\infty} \frac{1}{k^n} \int_0^z t^{k-1}\, dt = \sum_{k=1}^{\infty} \frac{z^k}{k^{n+1}} = \mathrm{Li}_{n+1}(z)$$

8.4 Square formula

$$\operatorname{Li}_n(-z) + \operatorname{Li}_n(z) = 2^{1-n} \operatorname{Li}_n(z^2)$$

proof

As usual we write the series representation of the LHS

$$\sum_{k=1}^{\infty} \frac{z^k}{k^n} + \sum_{k=1}^{\infty} \frac{(-z)^k}{k^n}$$

Listing the first few terms

$$z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots + \left(-z + \frac{z^2}{2^n} - \frac{z^3}{3^n} + \cdots\right)$$

The odd terms will cancel

$$2\frac{z^2}{2^n} + 2\frac{z^4}{4^n} + 2\frac{z^6}{6^n} + \cdots$$

Take 2^{1-n} as a common factor

$$2^{1-n}\left(z^2 + \frac{(z^2)^2}{2^n} + \frac{(z^2)^3}{3^n} + \cdots\right) = 2^{1-n} \sum_{k=1}^{\infty} \frac{(z^2)^k}{k^n} = 2^{1-n} \operatorname{Li}_n(z^2)$$

8.5 Exercise

Prove that

$$\operatorname{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\, dt$$

8.6 Dilogarithms

Of all polylogarithms $\mathrm{Li}_2(z)$ is the most interesting one, in this section we will see why!

8.6.1 Definition

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = -\int_0^z \frac{\log(1-t)}{t}\,dt$$

The curious reader should try to prove the integral representation using the recursive definition we introduced in the previous section .

8.6.2 First functional equation

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) + \mathrm{Li}_2(-z) = -\frac{1}{2}\log^2(z) - \frac{\pi^2}{6}$$

proof

We will start by the following

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) = -\int_0^{\frac{-1}{z}} \frac{\log(1-t)}{t}\,dt$$

Differentiate with respect to z

$$\frac{d}{dz}\mathrm{Li}_2\left(\frac{-1}{z}\right) = \frac{1}{z^2}\left(-\frac{\log\left(1+\frac{1}{z}\right)}{\frac{-1}{z}}\right) = \frac{\log\left(1+\frac{1}{z}\right)}{z} = \frac{\log(1+z)-\log(z)}{z}$$

Now integrate with respect to z

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) = \int_0^{-z} \frac{\log(1-t)}{t}\,dt - \frac{1}{2}\log^2(z) \;+C = -\mathrm{Li}_2(-z) - \frac{1}{2}\log^2(z) \;+C$$

To find the constant C let $z = 1$

$$C = 2\mathrm{Li}_2\left(-1\right)$$

Now we must be aware that

$$C = 2\mathrm{Li}_2(-1) = -2\eta(2) = \frac{-\pi^2}{6}$$

Which proves the result by simple rearrangement

$$\mathrm{Li}_2\left(\frac{-1}{z}\right) + \mathrm{Li}_2(-z) = -\frac{1}{2}\log^2(z) - \frac{\pi^2}{6}$$

8.6.3 Second functional equation

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z) \quad , 0 < z < 1$$

proof

Start by the following

$$\mathrm{Li}_2\,(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt$$

Now integrate by parts to obtain

$$\mathrm{Li}_2\,(z) = -\int_0^z \frac{\log(t)}{1-t}\,dt - \log(z)\log(1-z)$$

By the change of variable $t = 1-x$ we get

$$\int_0^z \frac{\log(t)}{1-t}\,dt \;= -\int_1^{1-z} \frac{\log(1-x)}{x}\,dx$$

For $0 < z < 1$

$$\int_{1-z}^1 \frac{\log(1-x)}{x}\,dx = \int_0^1 \frac{\log(1-x)}{x}\,dx - \int_0^{1-z} \frac{\log(1-x)}{x}\,dx$$

Now it is easy to see that

$$\int_{1-z}^1 \frac{\log(1-x)}{x}\,dx = -\mathrm{Li}_2(1) + \mathrm{Li}_2(1-z)$$

Which implies that

$$\mathrm{Li}_2\,(z) = \mathrm{Li}_2(1) - \mathrm{Li}_2(1-z) - \log(z)\log(1-z)$$

$$\mathrm{Li}_2\,(z) + \mathrm{Li}_2(1-z) \;=\; \mathrm{Li}_2(1) - \log(z)\log(1-z)$$

Now since $\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$

$$\mathrm{Li}_2\,(z) + \mathrm{Li}_2(1-z) \;=\; \frac{\pi^2}{6} - \log(z)\log(1-z)$$

We can easily deduce that for $z = \frac{1}{2}$

$$2\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{6} - \log^2\left(\frac{1}{2}\right)$$

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2\left(\frac{1}{2}\right)$$

8.6.4 Third functional equation

$$\operatorname{Li}_2(z) + \operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\log^2(1-z) \quad z < 1$$

proof

Start by the following

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\int_0^{\frac{z}{z-1}} \frac{\log(1-t)}{t}\,dt$$

Differentiate both sides with respect to z

$$\frac{d}{dz}\operatorname{Li}_2\left(\frac{z}{z-1}\right) = \frac{1}{(z-1)^2}\left(\frac{\log\left(1-\frac{z}{z-1}\right)}{\frac{z}{z-1}}\right)$$

Upon simplification we obtain

$$\frac{d}{dz}\operatorname{Li}_2\left(\frac{z}{z-1}\right) = \frac{-\log(1-z)}{z(z-1)}$$

Using partial fractions decomposition

$$\frac{d}{dz}\operatorname{Li}_2\left(\frac{z}{z-1}\right) = \frac{\log(1-z)}{1-z} + \frac{\log(1-z)}{z}$$

Integrate both sides with respect to z

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\log^2(1-z) - \operatorname{Li}_2(z) + C$$

Put $z = -1$ to find the constant

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = -\frac{1}{2}\log^2(2) - \operatorname{Li}_2(-1) + C$$

Remember that

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2\left(\frac{1}{2}\right)\ ,\ \operatorname{Li}_2(-1) = -\frac{\pi^2}{12}$$

Hence we deduce that $C = 0$

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) = -\frac{1}{2}\log^2(1-z) - \operatorname{Li}_2(z)$$

Which can be written as

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) + \operatorname{Li}_2(z) = -\frac{1}{2}\log^2(1-z)$$

8.6.5 Example

Prove that

$$\operatorname{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{10} - \log^2\left(\frac{\sqrt{5}-1}{2}\right)$$

proof

First we add the two functional equations of this section to obtain

$$\operatorname{Li}_2\left(\frac{z}{z-1}\right) + \frac{1}{2}\operatorname{Li}_2(z^2) - \operatorname{Li}_2(-z) = -\frac{1}{2}\log^2(1-z)$$

Now let $z = \frac{1-\sqrt{5}}{2}$

$$z^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{3-\sqrt{5}}{2} \implies \frac{z}{z-1} = \frac{\sqrt{5}-1}{1+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$$

Hence we have

$$\frac{3}{2}\operatorname{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) - \operatorname{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = -\frac{1}{2}\log^2\left(\frac{\sqrt{5}+1}{2}\right)$$

We already established the following functional equation

$$\operatorname{Li}_2(z) + \operatorname{Li}_2(1-z) = \frac{\pi^2}{6} - \log(z)\log(1-z)$$

Put $z = \frac{3-\sqrt{5}}{2}$

$$\operatorname{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) + \operatorname{Li}_2\left(\frac{\sqrt{5}-1}{2}\right) = \frac{\pi^2}{6} - \log\left(\frac{3-\sqrt{5}}{2}\right)\log\left(\frac{\sqrt{5}-1}{2}\right)$$

Solving the two representations for $\operatorname{Li}_2\left(\frac{\sqrt{5}-1}{2}\right)$ we get our result.

8.6.6 Example

Find the following integral

$$I = \int_0^1 \frac{\log(1-x)\log(x)}{x}\,dx$$

Integrate by parts

$$I = -\log(x)\operatorname{Li}_2(x)|_0^1 + \int_0^1 \frac{\operatorname{Li}_2(t)}{t}\,dt = \operatorname{Li}_3(1) = \sum_{k=1}^{\infty}\frac{1}{k^3} = \zeta(3)$$

8.6.7 Example

Evaluate the following integral

$$\int_0^x \frac{\log^2(1-t)}{t}\, dt \quad, 0 < x < 1$$

Integrating by parts we get the following

$$\int_0^x \frac{\log^2(1-t)}{t}\, dt = -\log(1-x)\operatorname{Li}_2(x) - \int_0^x \frac{\operatorname{Li}_2(t)}{1-t}\, dt$$

Now we are left with the following integral

$$\int_0^x \frac{\operatorname{Li}_2(t)}{1-t}\, dt = \int_{1-x}^1 \frac{\operatorname{Li}_2(1-t)}{t}\, dt$$

Using the first functional equation

$$\int_{1-x}^1 \frac{\frac{\pi^2}{6} - \operatorname{Li}_2(t) - \log(1-t)\log(t)}{t}\, dt$$

$$-\frac{\pi^2}{6}\log(1-x) - \int_{1-x}^1 \frac{\operatorname{Li}_2(t)}{t}\, dt - \int_{1-x}^1 \frac{\log(1-t)\log(t)}{t}\, dt$$

The first integral

$$\int_{1-x}^1 \frac{\operatorname{Li}_2(t)}{t}\, dt = \operatorname{Li}_3(1) - \operatorname{Li}_3(1-x)$$

The second integral is the same as the first exercise

$$\int_{1-x}^1 \frac{\log(1-t)\log(t)}{t} = \operatorname{Li}_3(1) + \log(1-x)\operatorname{Li}_2(1-x) - \operatorname{Li}_3(1-x)$$

Collecting the results together we obtain

$$\int_0^x \frac{\operatorname{Li}_2(t)}{1-t}\, dt = -\frac{\pi^2}{6}\log(1-x) - \operatorname{Li}_2(1-x)\log(1-x) + 2\operatorname{Li}_3(1-x) - 2\zeta(3)$$

Finally we have

$$\int_0^x \frac{\log^2(1-t)}{t}\, dt = -\log(1-x)\operatorname{Li}_2(x) + \frac{\pi^2}{6}\log(1-x) + \operatorname{Li}_2(1-x)\log(1-x) - 2\operatorname{Li}_3(1-x) + 2\zeta(3)$$

8.6.8 Example

Find the following integral

$$I(a) = \int_0^a \frac{x}{e^x - 1}\, dx$$

Start by the power expansion of $\frac{1}{1-e^{-x}}$

$$I(a) = \int_0^a \frac{x}{e^x - 1}\, dx = \int_0^a xe^{-x} \sum_{n=0}^{\infty} e^{-nx}$$

By swapping the integral and the summation

$$I(a) = \sum_{n=0}^{\infty} \int_0^a xe^{-x(n+1)}\, dx$$

The integral could be solved by parts

$$I(a) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} - \frac{e^{-(n+1)a}}{(n+1)^2} - \frac{ae^{-(n+1)a}}{(n+1)}$$

Distribute the summation to obtain

$$I(a) = \zeta(2) + a\log(1 - e^{-a}) - \mathrm{Li}_2(e^{-a})$$

9 Ordinary Hypergeometric function

Ordinary or sometimes called the Gauss hypergoemtric function is a generalization of the power expansion definition. Before we start with the definition we will explain some notations.

9.1 Definition

Define the raising factorial as follows

$$(z)_n = \begin{cases} 1 & : n = 0 \\ \frac{\Gamma(z+n)}{\Gamma(z)} & : n > 0 \end{cases}$$

Using this definition have

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

9.2 Some expansions using the hypergeomtric function

We can represent famous functions using the hypergeomtric function

1. Logarithm

$$z{}_2F_1(1,1;2;-z) = z\sum_{n=0}^{\infty} \frac{(1)_n(1)_n}{(2)_n} \frac{(-z)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n \frac{n!}{(n+1)!} z^{n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{z^{n+1}}{n+1} = \log(1+z)$$

2. Power function

$${}_2F_1(a,1;1;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(1)_n}{(1)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n = (1-z)^{-a}$$

3. Sine inverse

$$z\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2\right) = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} \frac{z^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{2n+1} \frac{z^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!\,(2n+1)} z^{2n+1}$$

Which can be written as

$$z\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n\,(n!)^2\,(2n+1)} z^{2n+1} = \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{4^n\,(2n+1)} z^{2n+1} = \arcsin(z)$$

Now we consider converting the Taylor expansion into the equivalent hypergeomtric representation Suppose the following

$$
{}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} t_k \, z^k \ \ , \ \ t_0 = 1
$$

Now consider the ratio

$$
\frac{t_{k+1}}{t_k} = \frac{(k+a)(k+b)}{(k+c)(k+1)} z
$$

Using this definition, we can easily find the terms a, b, c.

Let us consider some examples

1. Exponential function

$$
f(z) = e^z
$$

   The power expansion is

$$
f(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}
$$

   Hence we have

$$
\frac{t_{k+1}}{t_k} = \frac{z}{k+1}
$$

   Comparing to our representation we conclude

$$
e^z = {}_2F_1(-,-;-;z)
$$

2. Cosine function

$$
f(z) = \cos(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}
$$

   By the same approach

$$
\frac{t_{k+1}}{t_k} = -\frac{1}{(2k+2)(2k+1)} z = \frac{1}{(k+1)\left(k+\frac{1}{2}\right)} \frac{-z^2}{4}
$$

   Hence we have

$$
\cos(z) = {}_2F_1\left(-,-;\frac{1}{2};\frac{-z^2}{4}\right)
$$

3. Power function

$$f(z) = (1-z)^{-a} = \sum_{k=0}^{\infty} \frac{(a)_k}{k!} z^k$$

By the same approach

$$\frac{t_{k+1}}{t_k} = \frac{(k+a)}{(k+1)} z = \frac{(k+a)(k+1)}{(k+1)(k+1)} z$$

Hence we have

$$(1-z)^{-a} = {}_2F_1(a, 1; 1; z)$$

9.3 Exercise

Find the hypergeoemtric representations of the following functions

$$\arcsin(z), \sin(z)$$

9.4 Integral representation

$$\beta(c-b, b)\, {}_2F_1(a, b; c; z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\, dt$$

proof

Start by the RHS

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}\,(1-tz)^{-a}\, dt$$

Using the expansion of $(1-tz)^{-a}$ we have

$$\int_0^1 t^{b-1}(1-t)^{c-b-1} \sum_{k=0}^{\infty} \frac{(a)_k}{k!}\,(tz)^k$$

Interchanging the integral with the series

$$\sum_{k=0}^{\infty} \frac{(a)_k}{k!}\, z^k \int_0^1 t^{k+b-1}(1-t)^{c-b-1}\, dt$$

Recalling the beta function we have

$$\sum_{n=0}^{\infty} \frac{(a)_k \Gamma(k+b)\Gamma(c-b)}{\Gamma(k+c)}\, \frac{z^k}{k!}$$

Using the identity that

$$\beta(c-b,b) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}$$

and

$$\frac{\Gamma(z+k)}{\Gamma(z)} = (z)_k$$

We deduce that

$$\beta(c-b,b)\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(k+b)\Gamma(c)}{\Gamma(b)\Gamma(k+c)}\frac{z^k}{k!} = \beta(c-b,b)\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

9.5 Transformations

1. Pfaff transformations

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

and

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{-b}{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right)$$

proof

Start by the integral representation

$${}_2F_1\left(a,b;c;z\right) = \frac{1}{\beta(b,c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,dt$$

By the the transformation $t \to 1-t$

$$\int_0^1 \frac{(1-t)^{b-1}t^{c-b-1}}{(1-(1-t)z)^a}\,dt = \int_0^1 \frac{(1-t)^{b-1}t^{c-b-1}}{(1-z+tz)^a}\,dt$$

Which can be written as

$$\frac{(1-z)^{-a}}{\beta(b,c-b)}\int_0^1 t^{c-b-1}(1-t)^{b-1}\left(1-t\frac{z}{z-1}\right)^{-a}\,dt$$

Note this is the integral representation of

$$(1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

Also using that

$${}_2F_1\left(a,b;c;z\right) = {}_2F_1\left(b,a;c;z\right)$$

We deduce that

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{-b}{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right)$$

2. Euler transformation

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{c-a-b}{}_2F_1\left(c-a,c-b;c;z\right)$$

proof

In the Pfaff transformations let $z \to \frac{z}{z-1}$

$${}_2F_1\left(a,b;c;\frac{z}{z-1}\right) = (1-z)^{-a}{}_2F_1\left(a,c-b;c;z\right)$$

and

$${}_2F_1\left(a,b;c;\frac{z}{z-1}\right) = (1-z)^{-b}{}_2F_1\left(c-a,b;c;z\right)$$

By equating the two transformations

$$(1-z)^{-a}{}_2F_1\left(a,c-b;c;z\right) = (1-z)^{-b}{}_2F_1\left(c-a,b;c;z\right)$$

Now use the transformation $b \to c-b$

$$(1-z)^{-a}{}_2F_1\left(a,b;c;z\right) = (1-z)^{c-b}{}_2F_1\left(c-a,c-b;c;z\right)$$

Which can be reduced to

$${}_2F_1\left(a,b;c;z\right) = (1-z)^{c-a-b}{}_2F_1\left(c-a,c-b;c;z\right)$$

3. Quadratic transformation

$$_2F_1\left(a,b;2b;z\right) = (1-z)^{-\frac{a}{2}}\,{}_2F_1\left(\frac{a}{2}, b-\frac{a}{2}; b+\frac{1}{2}; \frac{z^2}{4z-4}\right)$$

4. Kummer

$$_2F_1\left(a,b;c;z\right) = {}_2F_1\left(a,b;1+a+b-c;1-z\right)$$

9.6 Special values

1. At $z = 1$

$$_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)}$$

Start by the integral representation at $z = 1$

$$_2F_1(a,b;c;1) = \frac{1}{\beta(c-b,b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,dt$$

Now we can use the first integral representation of the beta function

$$_2F_1(a,b;c;1) = \frac{1}{\beta(c-b,b)}\cdot\frac{\Gamma(b)\Gamma(c-b-a)}{\Gamma(c-a)}$$

Which could be simplified to

$$_2F_1(a,b;c;1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\cdot\frac{\Gamma(b)\Gamma(c-b-a)}{\Gamma(c-a)} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-a)}$$

2. At $z = -1$

$$_2F_1(a,b;1+a-b;-1) = \frac{\Gamma(1+a-b)\Gamma\left(1+\frac{a}{2}\right)}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)}$$

10 Error Function

The error function is an interesting function that has many applications in probability, statistics and physics.

10.1 Definition

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

10.2 Complementary error function

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

10.3 Imaginary error function

$$\operatorname{erfi}(x) = -i\operatorname{erf}(ix)$$

10.4 Properties

1. The error function is odd

$$\operatorname{erf}(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\, dt = -\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2}\, dt = -\operatorname{erf}(x)$$

2. Real part and imaginary parts

$$\Re \operatorname{erf}(z) = \frac{\operatorname{erf}(z) + \operatorname{erf}(\bar{z})}{2}$$

$$\Im \operatorname{erf}(z) = \frac{\operatorname{erf}(z) - \operatorname{erf}(\bar{z})}{2i}$$

Using complex variables it can be done using $\operatorname{erf}(\bar{z}) = \overline{\operatorname{erf}(z)}$

10.5 Relation to other functions

1. Hypergeomtric function

$$\operatorname{erf}(x) = \frac{2x}{\sqrt{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x^2\right)$$

proof

By expanding the hypergeometirc function

$$\frac{2x}{\sqrt{\pi}}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2},-x^2\right) = \frac{2x}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k}\frac{(-x^2)^k}{k!}$$

Which can be simplified to

$$\frac{2x}{\sqrt{\pi}}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2},-x^2\right) = \frac{x}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-x^2)^k}{(\frac{1}{2}+k)k!}$$

Notice that this is actually the expanded error function

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt = \frac{2}{\sqrt{\pi}}\int_0^x\sum_{k=0}^{\infty}\frac{(-x^2)^k}{k!}\,dt = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-x)^{2k+1}}{(2k+1)k!}$$

2. Incomplete Gamma function

$$\operatorname{erf}(x) = 1 - \frac{\Gamma\left(\frac{1}{2},x^2\right)}{\sqrt{\pi}}$$

proof

By the definition of the incomplete gamma function

$$\Gamma\left(\frac{1}{2},x^2\right) = \int_{x^2}^{\infty} t^{-\frac{1}{2}}\,e^{-t}\,dt$$

Let $t = y^2$

$$\Gamma\left(\frac{1}{2},x^2\right) = 2\int_x^{\infty} e^{-y^2}\,dy$$

We need to get the interval $(0,x)$

$$\Gamma\left(\frac{1}{2},x^2\right) = 2\int_0^{\infty} e^{-y^2}\,dy - 2\int_0^{x} e^{-y^2}\,dy$$

We have already proved that

$$2\int_0^{\infty} e^{-y^2}\,dy = \sqrt{\pi}$$

Hence we have using the definition of the error function

$$\Gamma\left(\frac{1}{2}, x^2\right) = \sqrt{\pi} - \sqrt{\pi}\text{erf}(x)$$

By rearrangements we get our result.

10.6 Example

Find the integral

$$I = \int_0^x e^{t^2}\, dt$$

The function has no elementary anti-derivative so we represent it using the error function.

Consider the imaginary error function

$$\text{erfi}(x) = -i\frac{2}{\sqrt{\pi}}\int_0^{ix} e^{-t^2}\, dt$$

By differentiating both sides we have

$$\frac{d}{dx}\text{erfi}(x) = \frac{2}{\sqrt{\pi}}e^{x^2}$$

Hence we have

$$e^{x^2} = \frac{d}{dx}\frac{\sqrt{\pi}}{2}\text{erfi}(x)$$

By integrating both sides we have

$$\int_0^x e^{t^2}\, dt = \frac{\sqrt{\pi}}{2}\text{erfi}(x)$$

10.7 Example

Prove that

$$\int_0^\infty \text{erfc}(x)\, dx = \frac{1}{\sqrt{\pi}}$$

proof

Using the complementary error function

$$\int_0^\infty (1 - \text{erf}(x))\, dx$$

Integrating by parts we have

$$I = x(1 - \text{erf}(x))\ ]_0^\infty + \frac{2}{\sqrt{\pi}} \int_0^\infty xe^{-x^2}\, dx$$

Now we compute erf(∞)

$$\text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1$$

So the first term will go to zero. The integral can be solved by substitution

$$I = \frac{2}{\sqrt{\pi}} \int_0^\infty xe^{-x^2}\, dx = \frac{1}{\sqrt{\pi}}$$

10.8 Example

Prove that

$$\int_0^\infty \text{erfc}^2(x)\, dx = \frac{2 - \sqrt{2}}{\sqrt{\pi}}$$

proof

Integrate by parts

$$I = x\text{erfc}^2(x)\ ]_0^\infty - 2\int_0^\infty x\text{erfc}'(x)\text{erfc}(x)\, dx$$

The first integral goes to 0

$$I = -2\int_0^\infty x\, \text{erfc}'(x)\text{erfc}(x)\, dx$$

The derivative of the complementary error function

$$\text{erfc}'(x) = (1 - \text{erf}(x))' = -\frac{2}{\sqrt{\pi}} e^{-x^2}$$

That results in

$$I = \frac{4}{\sqrt{\pi}} \int_0^\infty xe^{-x^2} \text{erfc}(x)\, dx$$

Integrate by parts again

$$I = \frac{2}{\sqrt{\pi}} e^{-x^2} \text{erfc}(x)\ ]_0^\infty - \frac{4}{\pi} \int_0^\infty e^{-2t^2}\, dt$$

At infinity the integral goes to 0. At 0 we get

$$\frac{2}{\sqrt{\pi}}e^{-x^2}\text{erfc}(x)]_{x=0} = \frac{2}{\sqrt{\pi}}(1-\text{erf}(0)) = \frac{2}{\sqrt{\pi}}$$

The integral can be evaluated to

$$\int_0^\infty e^{-2t^2}\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Collecting the results together we have

$$I = \frac{2}{\sqrt{\pi}} - \frac{\sqrt{\pi}}{2\sqrt{2}} \times \frac{4}{\pi} = \frac{2-\sqrt{2}}{\sqrt{\pi}}$$

10.9 Example

Prove that

$$\int_0^\infty \sin(x^2)\text{erfc}(x)\,dx = \frac{\pi - 2\coth^{-1}\sqrt{2}}{4\sqrt{2\pi}}$$

proof

Using the substitution $x = \sqrt{t}$

$$\frac{1}{2}\int_0^\infty \sin(t)t^{-\frac{1}{2}}\text{erfc}(\sqrt{t})\,dt$$

Consider the function

$$I(a) = \frac{1}{2}\int_0^\infty \sin(t)t^{-\frac{1}{2}}\text{erfc}(a\sqrt{t})\,dt$$

Differentiating with respect to a we have

$$I'(a) = \frac{-1}{\sqrt{\pi}}\int_0^\infty \sin(t)e^{-a^2 t}\,dt = \frac{-1}{\sqrt{\pi}}\cdot\frac{1}{a^4+1}$$

Now integrating with respect to a

$$I(a) = \frac{-1}{\sqrt{\pi}}\int_0^a \frac{dx}{x^4+1} + C$$

To evaluate the constant we take $a \to \infty$

$$I(\infty) = \frac{-1}{\sqrt{\pi}}\int_0^\infty \frac{dx}{x^4+1} + C$$

The function has an anti-derivative and the value is

$$\frac{-1}{\sqrt{\pi}}\int_0^\infty \frac{dx}{x^4+1} = -\frac{\sqrt{\pi}}{2\sqrt{2}}$$

Note that

$$\operatorname{erfc}(\infty) = 0 \quad \Longrightarrow \quad C = \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Finally we get

$$I(a) = \frac{-1}{\sqrt{\pi}}\int_0^a \frac{dx}{x^4+1} + \frac{\sqrt{\pi}}{2\sqrt{2}}$$

Let $a = 1$ in the integral

$$I(1) = \int_0^\infty \sin(x^2)\operatorname{erfc}(x)\,dx = \frac{\sqrt{\pi}}{2\sqrt{2}} - \frac{1}{\sqrt{\pi}}\int_0^1 \frac{dx}{x^4+1}$$

Also knowing that

$$\frac{1}{\sqrt{\pi}}\int_0^1 \frac{dx}{x^4+1} = \frac{\pi + 2\coth^{-1}\sqrt{2}}{4\sqrt{2\pi}}$$

Hence we have the result

$$\int_0^\infty \sin(x^2)\operatorname{erfc}(x)\,dx = \frac{\pi - 2\coth^{-1}\sqrt{2}}{4\sqrt{2\pi}}$$

10.10 Exercise

Can you find closed forms for

$$\int_0^\infty \operatorname{erfc}^3(x)\,dx = ?$$

$$\int_0^\infty \operatorname{erfc}^4(x)\,dx = ?$$

What about

$$\int_0^\infty \operatorname{erfc}^n(x)\,dx = ?$$

11 Exponential integral function

11.1 Definition

$$E(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = \int_1^\infty \frac{e^{-xt}}{t}\,dt$$

11.2 Example

Prove that

$$\lim_{x\to 0}\left[\log(x) + E(x)\right] = -\gamma$$

proof

Integration by parts for $E(x)$

$$E(x) = e^{-t}\log(t)\,\Big]_x^\infty + \int_x^\infty \log(t)e^{-t}\,dt$$

The limit at infinity goes to zero

$$E(x) = -e^{-x}\log(x) + \int_x^\infty \log(t)e^{-t}\,dt$$

Hence by taking the limit

$$\lim_{x\to 0}\left[\log(x) + E(x)\right] = \lim_{x\to 0}\left(\log(x) - e^{-x}\log(x)\right) + \int_0^\infty \log(t)e^{-t}\,dt$$

The first limit goes to 0

$$\lim_{x\to 0}\left[\log(x) + E(x)\right] = \int_0^\infty \log(t)e^{-t}\,dt = \psi(1) = -\gamma$$

11.3 Example

Prove that for $p > 0$

$$\int_0^\infty x^{p-1}E(ax)\,dx = \frac{\Gamma(p)}{pa^p}$$

proof

Integrating by parts we have

$$\int_0^\infty x^{p-1} E(ax)\, dx = \frac{1}{p} x^p E(ax)\ \Big]_0^\infty + \frac{1}{ap} \int_0^\infty x^{p-1} e^{-ax}\, dx$$

The first limit goes to 0

$$\frac{1}{ap} \int_0^\infty x^{p-1} e^{-ax}\, dx = \frac{1}{pa^p} \int_0^\infty x^{p-1} e^{-x}\, dx = \frac{\Gamma(p)}{p\, a^p}$$

11.4 Example

Prove the general case

$$\int_0^\infty x^{p-1} e^{ax} E(ax)\, dx = \frac{\pi}{\sin(a\pi)} \cdot \frac{\Gamma(p)}{a^p}$$

proof

Switch to the integral representation

$$\int_0^\infty x^{p-1} e^{ax} \int_{ax}^\infty \frac{e^{-t}}{t}\, dt\, dx$$

Use the substitution $t = ax\, y$

$$\int_0^\infty \int_1^\infty x^{p-1} \frac{e^{-ax(y-1)}}{y}\, dy\, dx$$

By switching the two integrals

$$\int_1^\infty \frac{1}{y} \int_0^\infty x^{p-1} e^{-ax(y-1)} dx\, dy$$

By the Laplace identities

$$\frac{\Gamma(p)}{a^p} \int_1^\infty \frac{1}{y(y-1)^p}\, dy$$

Now let $y = 1/x$

$$\frac{\Gamma(p)}{a^p} \int_0^1 x^{p-1} (1-x)^{-p}\, dx$$

Using the reflection formula for the Gamma function

$$\frac{\Gamma(p)}{a^p} \int_0^1 x^{p-1} (1-x)^{-p}\, dx = \frac{\pi}{\sin(a\pi)} \cdot \frac{\Gamma(p)}{a^p}$$

11.5 Example

Prove that

$$\int_0^\infty e^z E^2(z)\, dz = \frac{\pi^2}{6}$$

proof

Using the integral representation

$$E^2(z) = \int_1^\infty \int_1^\infty \frac{e^{-xz}e^{-yz}}{xy}\, dx\, dy$$

$$\int_0^\infty e^z E^2(z)\, dz = \int_0^\infty \int_1^\infty \int_1^\infty \frac{e^{-z(x+y-1)}}{xy}\, dx\, dy\, dz$$

Swap the integrals

$$\int_1^\infty \frac{1}{y} \int_1^\infty \frac{1}{x} \int_0^\infty e^{-z(x+y-1)}\, dz\, dx\, dy$$

$$\int_1^\infty \frac{1}{y} \int_1^\infty \frac{1}{x(x+y-1)}\, dx\, dy$$

The inner integral is an elementary integral

$$\int_1^\infty \frac{1}{x(x+y-1)}\, dx = -\frac{\log(y)}{1-y}$$

The integral becomes

$$\int_1^\infty \frac{\log(y)}{y(y-1)}\, dy$$

Now use the substitution $y = 1/x$

$$-\int_0^1 \frac{\log(x)}{(1-x)}\, dx = -\int_0^1 \frac{\log(1-x)}{x}\, dx = \mathrm{Li}_2(1) = \frac{\pi^2}{6}$$

11.6 Example

Prove that

$$\int_0^\infty z^{p-1} E^2(z)\, dz = \frac{2\Gamma(p)}{p^2}\, {}_2F_1(p, p; p+1; -1)$$

proof

Consider

$$E^2(z) = \int_z^\infty \frac{e^{-t}}{t}\,dt$$

By differentiation with respect to z

$$2E'(z)E(z) = \frac{2\,e^{-z}E(z)}{z}$$

Knowing that we can return to our integral by integration by parts

$$\frac{2}{p}\int_0^\infty z^{p-1}e^{-z}E(z)\,dz$$

Write the integral representation

$$\frac{2}{p}\int_0^\infty z^{p-1}e^{-z}\int_1^\infty \frac{e^{-zt}}{t}\,dtdz$$

Swap the two integrals

$$\frac{2}{p}\int_1^\infty \frac{1}{t}\int_0^\infty z^{p-1}e^{-z(1+t)}\,dz\,dt$$

The inner integral reduces to

$$\frac{2\Gamma(p)}{p}\int_1^\infty \frac{dt}{t(1+t)^p}$$

Use the substitution $t = 1/x$

$$\frac{2\Gamma(p)}{p}\int_0^1 \frac{x^{p-1}}{(1+x)^p}\,dx$$

Using the integral representation of the Hypergeometric function

$$\beta(c-b,b)\,{}_2F_1(a,b;c;z) = \int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-xz)^a}\,dx$$

Let $c = p+1,\ b = p,\ a = p,\ z = -1$

$$\beta(1,p)\,{}_2F_1(p,p;p+1;-1) = \int_0^1 \frac{x^{p-1}}{(1+x)^p}\,dx$$

Hence the result

$$\int_0^\infty z^{p-1}E^2(z)\,dz = \frac{2\Gamma(p)}{p^2}\,{}_2F_1(p,p;p+1;-1)$$

Where

$$\beta(1,p) = \frac{\Gamma(p)}{\Gamma(p+1)} = \frac{1}{p}$$

11.7 Exercise

Find the integral fo $n \in \mathbb{N}$

$$\int_0^\infty x^n E^2(x)\,dx$$

12 Complete Elliptic Integral

12.1 Complete elliptic of first kind

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

12.2 Complete elliptic of second kind

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx$$

12.3 Hypergeometric representation

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,k^2\right)$$

and

$$E(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2},1,k^2\right)$$

proof

Using the integral representation of the hypergeometric function

$$\beta(c-b,b)\,{}_2F_1(a,b,c,z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,dt$$

Now use the substitution $t = x^2$ and $z = k^2$

$$\beta(c-b,b)\,{}_2F_1(a,b,c,k^2) = 2\int_0^1 \frac{x^{2b-1}(1-x^2)^{c-b-1}}{(1-k^2x^2)^a}\,dx$$

Put $a = \frac{1}{2}$; $b = \frac{1}{2}$ and $c = 1$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} = \frac{1}{2}\beta(1/2,1/2)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,k^2\right)$$

By the beta function we have

$$\frac{1}{2}\beta(1/2,1/2) = \frac{1}{2}\Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{2}$$

Hence the result

$$K(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,k^2\right)$$

By the same approach we have

$$E(k) = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},-\frac{1}{2},1,k^2\right)$$

12.4 Example

Prove that

$$\int_0^1 K(k)\,dk = 2G$$

G is the Catalan's constant.

proof

Start by the integral representation

$$I = \int_0^1\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\,dx\,dk$$

Switching the two integrals

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}}\int_0^1 \frac{1}{\sqrt{1-k^2x^2}}\,dk\,dx$$

$$I = \int_0^1 \frac{\arcsin x}{x\sqrt{1-x^2}}dx$$

Now let $\arcsin x = t$ hence we have $x = \sin t$

$$I = \int_0^{\frac{\pi}{2}} \frac{t}{\sin t}dt$$

The previous integral is a representation of the constant

$$G = \frac{I}{2} \implies I = 2G$$

12.5 Identities

1. For $k \geq 1$

$$K(\sqrt{k}) = \frac{1}{\sqrt{1-k}}K\left(\sqrt{\frac{k}{k-1}}\right)$$

and

$$E(\sqrt{k}) = \sqrt{1-k}\, E\left(\sqrt{\frac{k}{k-1}}\right)$$

proof

Starting by the integral representation

$$K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

Use the substitution $x = \sqrt{1-y^2}$

$$\int_0^1 \frac{y\, dy}{\sqrt{1-y^2}\sqrt{1-(1-y^2)}\sqrt{1-k^2(1-y^2)}}$$

By cancelling the terms we have

$$\int_0^1 \frac{dy}{\sqrt{1-y^2}\sqrt{1-k^2+k^2y^2}}$$

Take $\sqrt{1-k^2}$ as a common factor

$$\int_0^1 \frac{dy}{\sqrt{1-k^2}\sqrt{1-y^2}\sqrt{1-\frac{k^2}{k^2-1}y^2}}$$

Comparing this to the integral representation we get

$$K(k) = \frac{1}{\sqrt{1-k^2}} K\left(\sqrt{\frac{k^2}{k^2-1}}\right)$$

We can finish by $k \to \sqrt{k}$

$$K(\sqrt{k}) = \frac{1}{\sqrt{1-k}} K\left(\sqrt{\frac{k}{k-1}}\right)$$

Similarly for the second representation

$$E(\sqrt{k}) = \int_0^1 \frac{\sqrt{1-kx^2}}{\sqrt{1-x^2}}\, dx$$

By using that $x = \sqrt{1-y^2}$

$$E(\sqrt{k}) = \sqrt{1-k} \int_0^1 \frac{\sqrt{1-\frac{k}{k-1}y}}{\sqrt{1-y^2}}\, dy = \sqrt{1-k}\, E\left(\sqrt{\frac{k}{k-1}}\right)$$

2.

$$K(k) = \frac{1}{k+1} K\left(\frac{2\sqrt{k}}{1+k}\right)$$

proof

Start by the Quadratic transformation

$${}_2F_1\left(a, b, 2b, \frac{4z}{(1+z)^2}\right) = (1+z)^{2a}\,{}_2F_1\left(a, a-b+\frac{1}{2}, b+\frac{1}{2}, z^2\right).$$

Hence we can deduce by putting $a = b = 1/2$

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)K(k)$$

Or we have

$$K(k) = \frac{1}{k+1} K\left(\frac{2\sqrt{k}}{1+k}\right)$$

3.

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1+k}{1-k} K\left(\frac{2\sqrt{-k}}{1-k}\right)$$

and

$$E\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1-k}{1+k} E\left(\frac{2\sqrt{-k}}{1-k}\right)$$

proof

Start by the following

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{1-\frac{4k}{(1+k)^2}x^2}}\,dx$$

By some simplifications we have

$$E\left(\frac{2\sqrt{k}}{1+k}\right) = (1+k)\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{(1+k)^2-4k\,x^2}}\,dx$$

Use $x = \sqrt{1-y^2}$

$$\int_0^1 \frac{1+k}{\sqrt{1-y^2}\sqrt{(1+k)^2-4k\,(1-y^2)}}\,dy = \frac{1+k}{1-k}\int_0^1 \frac{1}{\sqrt{1-y^2}\sqrt{1+\frac{4k}{(1-k)^2}y^2}}\,dy$$

Hence we have

$$K\left(\frac{2\sqrt{k}}{1+k}\right)=\frac{1+k}{1-k}K\left(\frac{2\sqrt{-k}}{1-k}\right)$$

Similarly we have

$$E\left(\frac{2\sqrt{k}}{1+k}\right)=\frac{1-k}{1+k}E\left(\frac{2\sqrt{-k}}{1-k}\right)$$

12.6 Special values

1.

$$K(i)=\frac{1}{4\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

and

$$E(i)=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}}+\frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

proof

By definition we have

$$K(i)=\int_0^1\frac{dx}{\sqrt{1-x^2}\sqrt{1+x^2}}=\int_0^1\frac{dx}{\sqrt{1-x^4}}$$

Let $x=\sqrt[4]{t}$ we have $dx=\frac{1}{4}t^{\frac{-3}{4}}\,dt$

$$K(i)=\frac{1}{4}\int_0^1 t^{\frac{-3}{4}}(1-t)^{\frac{-1}{2}}\,dt$$

By beta function

$$K(i)=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{3}{4}\right)}$$

By reflection formula

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\pi\csc\left(\frac{\pi}{4}\right)=\pi\sqrt{2}$$

$$K(i)=\frac{\Gamma^2\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\pi\sqrt{2}}=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}}$$

$$E(i) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

By definition we have

$$E(i) = \int_0^1 \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} dx$$

Separating the two integrals

$$E(i) = \int_0^1 \frac{1+x^2}{\sqrt{1-x^4}} dx = \int_0^1 \frac{1}{\sqrt{1-x^4}} dx + \int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx$$

The first integral is $K(i)$ for the second integral use $x = \sqrt[4]{t}$

$$\frac{1}{4}\int_0^1 t^{\frac{3}{4}-1}(1-t)^{-\frac{1}{2}}\, dt = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{4\Gamma\left(\frac{5}{4}\right)} = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

Hence we have

$$E(i) = K(i) + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi}}$$

2.

$$K\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

and

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{2\sqrt{\pi}}$$

proof

Start by the identity

$$K(\sqrt{k}) = \frac{1}{\sqrt{1-k}} K\left(\sqrt{\frac{k}{k-1}}\right)$$

For the value $k = -1$

$$K(i) = \frac{1}{\sqrt{2}} K\left(\frac{1}{\sqrt{2}}\right)$$

Using the value for $K(i)$

$$K\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}K(i) = \frac{1}{4\sqrt{\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

Similarly we have

$$E\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{2\sqrt{\pi}}$$

3.

$$K\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{1+\sqrt{2}}{4\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

and

$$E\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{\sqrt{2}}{1+\sqrt{2}}\left[\frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}\right]$$

proof

Start by the identity

$$K(k) = \frac{1}{k+1}K\left(\frac{2\sqrt{k}}{1+k}\right)$$

Hence we have for $k = \frac{1}{\sqrt{2}}$

$$K\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{1+\sqrt{2}}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) = \frac{1+\sqrt{2}}{4\sqrt{2\pi}}\Gamma^2\left(\frac{1}{4}\right)$$

For the elliptic integral of second kind using the hypergeomtric representation with $a = \frac{-1}{2}$ and $b = \frac{1}{2}$

$${}_2F_1\left(-1/2, 1/2, 1, \frac{4z}{(1+z)^2}\right) = (1+z)^{-1}{}_2F_1\left(-1/2, -1/2, 1, z^2\right)$$

The later hypergeometric series can be written in terms of elliptic integrals using some general contiguity relations

$${}_2F_1\left(-1/2, -1/2, 1, z^2\right) = \frac{2}{\pi}\left(2E(k) + (k^2-1)K(k)\right)$$

So we have

$$2E(k) + (k^2-1)K(k) = (k+1)E\left(\frac{2\sqrt{k}}{1+k}\right)$$

For $k = \frac{1}{\sqrt{2}}$

$$E\left(2\sqrt{-4+3\sqrt{2}}\right) = \frac{\sqrt{2}}{1+\sqrt{2}}\left[\frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} + \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{\pi}}\right]$$

4.

$$K\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{\pi\sqrt{\pi}}{4}\cdot\frac{2-\sqrt{2}}{\Gamma^2\left(\frac{3}{4}\right)}$$

and

$$E\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{(2+\sqrt{2})\left(\pi^2+4\Gamma^4\left(\frac{3}{4}\right)\right)}{4\sqrt{\pi}\Gamma^2\left(\frac{3}{4}\right)}$$

proof

Start by the following identity

$$K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{1+k}{1-k}K\left(\frac{2\sqrt{-k}}{1-k}\right)$$

Let $x = 1/\sqrt{2}$

$$K\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{\pi\sqrt{\pi}}{4}\cdot\frac{2-\sqrt{2}}{\Gamma^2\left(\frac{3}{4}\right)}$$

$$E\left(2\sqrt{-4-3\sqrt{2}}\right) = \frac{(2+\sqrt{2})\left(\pi^2+4\Gamma^4\left(\frac{3}{4}\right)\right)}{4\sqrt{\pi}\Gamma^2\left(\frac{3}{4}\right)}$$

12.7 Differentiation of elliptic integrals

Note We should remove the variable k and denote elliptic integrals E and K once there is no confusion. It is assumed that the variable is k when we use these symbols.

Interestingly the derivative of elliptic integrals can be written in terms of elliptic integrals

Derivative of complete elliptic integral of second kind

$$\frac{d}{dk}E = \int_0^1 \frac{\frac{\partial}{\partial k}\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx$$

$$\frac{d}{dk}E = \int_0^1 \frac{-k\,x^2}{\sqrt{1-x^2}\sqrt{1-k^2\,x^2}}\,dx$$

Adding and subtracting 1 results in

$$\frac{1}{k}\int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}\,dx - \frac{1}{k}\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2\,x^2}}$$

Upon realizing the relation to elliptic integrals we conclude

$$\frac{d}{dk}E = \frac{E - K}{k}$$

For the complete elliptic integral of first kind we need more work

Start by the following

$$\frac{d}{dk}K = \int_0^1 \frac{1}{\sqrt{1-x^2}} \frac{\partial}{\partial k}\left[\frac{1}{\sqrt{1-k^2x^2}}\right] dx$$

$$\frac{d}{dk}K = \int_0^1 \frac{kx^2}{\sqrt{1-x^2}\sqrt{1-k^2x^2}(1-k^2x^2)}\, dx$$

Adding and subtracting 1 we have

$$\frac{-1}{k}\int_0^1 \frac{1-kx^2-1}{\sqrt{1-x^2}\sqrt{1-k^2x^2}(1-k^2x^2)}\, dx = \frac{1}{k}\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}(1-k^2x^2)} - \frac{K}{k}$$

Let us focus on the first integral

$$\int_0^1 \frac{1}{\sqrt{1-x^2}(1-\,k^2\,x^2)^{\frac{3}{2}}}\, dx$$

Let $x = \sqrt{t}$ and we have $dx = \frac{1}{2\sqrt{t}}\, dt$

$$\frac{1}{2}\int_0^1 \frac{t^{-\frac{1}{2}}}{\sqrt{1-t}(1-\,k^2\,t)^{\frac{3}{2}}}\, dx$$

Using the hypergeometric integral representation

$$\frac{1}{2}\int_0^1 \frac{t^{-\frac{1}{2}}}{\sqrt{1-t}(1-\,k^2\,t)^{\frac{3}{2}}} = \frac{\pi}{2}{}_2F_1\left(\frac{3}{2}, \frac{1}{2}, 1, k^2\right)$$

Using the linear transformation

$${}_2F_1\left(a, b, c, z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(c-a, c-b, c, z\right)$$

We get by putting $k' = \sqrt{1-k^2}$

$$\frac{\pi}{2}{}_2F_1\left(\frac{3}{2}, \frac{1}{2}, 1, k^2\right) = \frac{1}{1-k^2}\frac{\pi}{2}{}_2F_1\left(-\frac{1}{2}, \frac{1}{2}, 1, k^2\right) = \frac{E}{k'^2}$$

So finally we get

$$\frac{d}{dk}K = \frac{1}{k}\left(\frac{E}{k'^2} - K\right)$$

13 Euler sums

13.1 Definition

$$S_{p^r,q} = \sum_{k=1}^{\infty} \frac{(H_k^{(p)})^r}{k^q}$$

Where we define the general harmonic number

$$H_k^{(p)} = \sum_{n=1}^{k} \frac{1}{n^p} \;\; ; \;\; H_k^{(1)} \equiv H_k = \sum_{n=1}^{k} \frac{1}{n}$$

Euler sums were greatly studied by Euler, hence the name.

13.2 Generating function

$$\sum_{k=1}^{\infty} H_k^{(p)} x^k = \frac{\mathrm{Li}_p(x)}{1-x}$$

Proof

Start by writing $H_k^{(p)}$ as a sum

$$\sum_{k=1}^{\infty} H_k^{(p)} x^k = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{1}{n^p} x^k$$

By interchanging the two series we have

$$\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{x^k}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \sum_{k=n}^{\infty} x^k$$

The inner sum is a geometric series

$$\frac{1}{1-x} \sum_{n=1}^{\infty} \frac{x^n}{n^p} = \frac{\mathrm{Li}_p(x)}{1-x}$$

We can use this to generate some more functions by integrating.

13.3 Integral representation of Harmonic numbers

$$H_n = \int_0^1 \frac{1-x^n}{1-x} \, dx$$

proof

We can use the geometric series of x^n

$$\int_0^1 \frac{1-x^n}{1-x} \, dx = \sum_{k=0}^{\infty} \int_0^1 x^k - x^{n+k} \, dx$$

$$\sum_{k=0}^{\infty} \frac{1}{k+1} - \frac{1}{n+k+1} = H_n$$

13.4 Example

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3)$$

proof

Using the integral representation

$$\int_0^1 \frac{1}{1-x} \sum_{n=1}^{\infty} \frac{1-x^n}{n^2}\, dx = \int_0^1 \frac{\zeta(2) - \mathrm{Li}_2(x)}{1-x} dx$$

Now use the functional equation

$$\zeta(2) - \mathrm{Li}_2(x) = \mathrm{Li}_2(1-x) + \log(x)\log(1-x)$$

Hence we have

$$\int_0^1 \frac{\mathrm{Li}_2(1-x) + \log(x)\log(1-x)}{1-x} dx$$

The first integral

$$\int_0^1 \frac{\mathrm{Li}_2(1-x)}{1-x} = \mathrm{Li}_3(1) = \zeta(3)$$

The Second integral using integration by parts

$$\int_0^1 \frac{\log(1-x)\log(x)}{x} dx = \int_0^1 \frac{\mathrm{Li}_2(x)}{x} = \zeta(3)$$

Finally we have

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = \zeta(3) + \zeta(3) = 2\zeta(3)$$

13.5 Example

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} x^k = \mathrm{Li}_3(x) - \mathrm{Li}_3(1-x) + \log(1-x)\mathrm{Li}_2(1-x) + \frac{1}{2}\log(x)\log^2(1-x) + \zeta(3)$$

proof

In the general definition assume $p = 1$

$$\sum_{k=1}^{\infty} H_k x^k = -\frac{\log(1-x)}{1-x}$$

Divide by x and integrate to get

$$\sum_{k=1}^{\infty} \frac{H_k}{k} x^k = \mathrm{Li}_2(x) + \frac{1}{2}\log^2(1-x)$$

Now divide by x and integrate again

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} x^k = \mathrm{Li}_3(x) + \frac{1}{2}\int_0^x \frac{\log^2(1-t)}{t}\, dt$$

Now let us look at the integral

$$\int_0^x \frac{\log^2(1-t)}{t}\, dt$$

Integrating by parts

$$\int_0^x \frac{\log^2(1-t)}{t}\, dt = -\log(1-x)\mathrm{Li}_2(x) - \int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\, dt$$

Use a change of variable in the integral

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\, dt = \int_{1-x}^1 \frac{\mathrm{Li}_2(1-t)}{t}\, dt$$

Now we can use the second functional equation of the dilogarithm

$$\int_{1-x}^1 \frac{\frac{\pi^2}{6} - \mathrm{Li}_2(t) - \log(1-t)\log(t)}{t}\, dt$$

Separate the integrals

$$-\frac{\pi^2}{6}\log(1-x) - \int_{1-x}^1 \frac{\mathrm{Li}_2(t)}{t}\, dt - \int_{1-x}^1 \frac{\log(1-t)\log(t)}{t}\, dt$$

The first integral

$$\int_{1-x}^1 \frac{\mathrm{Li}_2(t)}{t}\, dt = \mathrm{Li}_3(1) - \mathrm{Li}_3(1-x)$$

Use integration by parts in the second integral

$$\int_{1-x}^1 \frac{\log(1-t)\log(t)}{t} = \mathrm{Li}_3(1) + \log(1-x)\mathrm{Li}_2(1-x) - \mathrm{Li}_3(1-x)$$

Collecting the results together we obtain

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}\,dt = -\frac{\pi^2}{6}\log(1-x) - \mathrm{Li}_2(1-x)\log(1-x) + 2\,\mathrm{Li}_3(1-x) - 2\zeta(3)$$

Hence we solved the integral

$$\int_0^x \frac{\log^2(1-t)}{t}\,dt = -\log(1-x)\mathrm{Li}_2(x) + \frac{\pi^2}{6}\log(1-x) + \mathrm{Li}_2(1-x)\log(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta(3)$$

So we have got our Harmonic sum

$$\sum_{k=1}^\infty \frac{H_k}{k^2}\,x^k = \mathrm{Li}_3(x) + \frac{1}{2}\left(-\log(1-x)\mathrm{Li}_2(x) + \frac{\pi^2}{6}\log(1-x) + \mathrm{Li}_2(1-x)\log(1-x) - 2\,\mathrm{Li}_3(1-x) + 2\zeta(3)\right)$$

$$\sum_{k=1}^\infty \frac{H_k}{k^2}\,x^k = \mathrm{Li}_3(x) - \mathrm{Li}_3(1-x) + \log(1-x)\mathrm{Li}_2(1-x) + \frac{1}{2}\log(x)\log^2(1-x) + \zeta(3)$$

13.6 General formula

$$\sum_{n=1}^\infty \frac{H_n}{n^q} = \left(1+\frac{q}{2}\right)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k)$$

This can be proved using complex analysis.

13.7 Example

$$\int_0^1 \frac{\log^2(1-x)\log(x)}{x} = -\frac{\pi^4}{180}$$

proof

Using the generating function

$$\sum_{k=1}^\infty H_k x^{k-1} = -\frac{\log(1-x)}{x(1-x)}$$

By integrating both sides

$$\sum_{k=1}^\infty \frac{H_k}{k}x^k = \mathrm{Li}_2(x) + \frac{1}{2}\log^2(1-x)$$

Or

$$\log^2(1-x) = 2\sum_{k=1}^\infty \frac{H_k}{k}x^k - 2\mathrm{Li}_2(x)$$

plugging this in our integral we have

$$2\int_0^1 \left(\sum_{k=1}^\infty \frac{H_k}{k}x^k - \operatorname{Li}_2(x)\right)\frac{\log(x)}{x}\,dx$$

Which simplifies to

$$2\sum_{k=1}^\infty \frac{H_k}{k}\int_0^1 \log(x)x^{k-1}\,dx - 2\int_0^1 \operatorname{Li}_2(x)\frac{\log(x)}{x}\,dx$$

The second integral

$$-2\int_0^1 \operatorname{Li}_2(x)\frac{\log(x)}{x}\,dx = 2\int_0^1 \frac{\operatorname{Li}_3(x)}{x}\,dx = 2\zeta(4)$$

The first integral

$$2\sum_{k=1}^\infty \frac{H_k}{k}\int_0^1 x^{k-1}\log(x)\,dx$$

Using integration by parts twice and the general formula for $q = 3$

$$-2\sum_{k=1}^\infty \frac{H_k}{k^3} = -5\zeta(4) + \zeta^2(2)$$

Finally we get

$$\int_0^1 \frac{\log^2(1-x)\log(x)}{x} = -5\zeta(4) + \zeta^2(2) + 2\zeta(4) = \zeta^2(2) - 3\zeta(4)$$

13.8 Example

Show that

$$\int_0^\infty e^{-at}\sin(bt)\frac{\log t}{t}\,dt = -\left(\frac{\log(a^2+b^2)}{2} + \gamma\right)\arctan\left(\frac{b}{a}\right)$$

Proof

We can start by the following integral

$$I(s) = \int_0^\infty t^{s-1}\,e^{-at}\,\sin(bt)dt$$

By using the the expansion of the sine function

$$I(s) = \int_0^\infty t^{s-1}\,e^{-at}\sum_{n=0}^\infty \frac{(-1)^n(bt)^{2n+1}}{\Gamma(2n+2)}$$

By swapping the summation and integration

$$I(s)=\sum_{n=0}^{\infty}\frac{(-1)^n(b)^{2n+1}}{\Gamma(2n+2)}\int_0^{\infty} t^{s+2n}\,e^{-at}dt=\frac{1}{a^s}\sum_{n=0}^{\infty}\frac{(-1)^n(b)^{2n+1}\Gamma(s+2n+1)}{\Gamma(2n+2)a^{2n+1}}$$

By differentiating and plugging $s=0$ we have

$$I'(0)=\sum_{n=0}^{\infty}\frac{(-1)^n\psi_0(2n+1)}{2n+1}\left(\frac{b}{a}\right)^{2n+1}-\log(a)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{b}{a}\right)^{2n+1}$$

Now use that $\psi(n+1)+\gamma=H_n$

$$I'(0)=\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n}-\gamma}{2n+1}\left(\frac{b}{a}\right)^{2n+1}-\log(a)\arctan\left(\frac{b}{a}\right)$$

$$I'(0)=\sum_{n=0}^{\infty}\frac{(-1)^nH_{2n}}{2n+1}\left(\frac{b}{a}\right)^{2n+1}-(\gamma+\log(a))\arctan\left(\frac{b}{a}\right)$$

Now we look at the harmonic sum

$$\sum_{k=0}^{\infty}(-1)^kH_{2k}x^{2k}=\sum_{k=0}^{\infty}(-1)^kx^{2k}\int_0^1\frac{1-t^{2k}}{1-t}\,dt$$

Use the integral representation

$$\int_0^1\frac{1}{1-t}\sum_{k=0}^{\infty}(-1)^kx^{2k}\left(1-t^{2k}\right)\,dt$$

Swap the series and the integral

$$\int_0^1\frac{1}{1-t}\sum_{k=0}^{\infty}(-1)^k\left(x^{2k}-(xt)^{2k}\right)\,dt$$

Evaluate the geometric series

$$\int_0^1\frac{1}{1-t}\left(\frac{1}{1+x^2}-\frac{1}{1+t^2x^2}\right)dt=\frac{-x^2}{1+x^2}\int_0^1\frac{(1-t^2)}{(1-t)(1+t^2x^2)}\,dt$$

which simplifies to

$$\frac{-x^2}{1+x^2}\int_0^1\frac{1+t}{(1+t^2x^2)}\,dt=\frac{-x^2}{1+x^2}\left(\int_0^1\frac{1}{1+t^2x^2}+\frac{t}{1+t^2x^2}\,dt\right)$$

Evaluating the integrals

$$\frac{-1}{2(1+x^2)}\left(2x\arctan(x)+\log(1+x^2)\right)$$

Using this we conclude by integrating

$$\sum_{k=0}^{\infty}\frac{(-1)^kH_{2k}}{2k+1}x^{2k}=-\frac{1}{2}\log(1+x^2)\arctan(x)$$

Hence the following

$$\sum_{k=0}^{\infty}\frac{(-1)^k H_{2k}}{2k+1}\left(\frac{b}{a}\right)^{2k+1} = -\frac{1}{2}\log\left(\frac{a^2+b^2}{a^2}\right)\arctan\left(\frac{b}{a}\right)$$

Substituting that in our integral

$$\begin{aligned}\int_0^{\infty} e^{-at}\sin(bt)\frac{\log t}{t}\,dt &= -\left(\frac{1}{2}\log\left(\frac{a^2+b^2}{a^2}\right)+\gamma+\log(a)\right)\arctan\left(\frac{b}{a}\right)\\ &= -\left(\frac{\log(a^2+b^2)}{2}+\gamma\right)\arctan\left(\frac{b}{a}\right)\end{aligned}$$

13.9 Example

$$\begin{aligned}\int_0^1 \frac{\mathrm{Li}_p(x)\,\mathrm{Li}_q(x)}{x}\,dx &= \sum_{n=1}^{p-1}(-1)^{n-1}\zeta(p-n+1)\zeta(q+n) - \frac{1}{2}\sum_{n=1}^{p+q-2}(-1)^{p-1}\zeta(n+1)\zeta(p+q-n)\\ &+ (-1)^{p-1}\left(1+\frac{p+q}{2}\right)\zeta(p+q+1)\end{aligned}$$

proof

We can see that

$$\int_0^1 \frac{\mathrm{Li}_p(x)\,\mathrm{Li}_q(x)}{x}\,dx = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^q n^p (n+k)}$$

Let us first look at the following

$$\mathscr{C}(\alpha,k) = \sum_{n=1}^{\infty}\frac{1}{n^{\alpha}(n+k)} \quad ; \quad \mathscr{C}(1,k) = \frac{H_k}{k}$$

This can be solved using

$$\begin{aligned}\mathscr{C}(\alpha,k) &= \sum_{n=1}^{\infty}\frac{1}{k\,n^{\alpha-1}}\left(\frac{1}{n}-\frac{1}{n+k}\right)\\ &= \frac{1}{k}\zeta(\alpha) - \frac{1}{k}\mathscr{C}(\alpha-1,k)\\ &= \frac{1}{k}\zeta(\alpha) - \frac{1}{k^2}\zeta(\alpha-1) + \frac{1}{k^2}\mathscr{C}(\alpha-2,k)\\ &.\\ &.\\ &.\\ &= \frac{1}{k}\zeta(\alpha) - \frac{1}{k^2}\zeta(\alpha-1) + \cdots + (-1)^{\alpha}\frac{\zeta(2)}{k^{\alpha-1}} + \frac{1}{k^{\alpha-1}}\mathscr{C}(\alpha-(\alpha-1),k)\\ &= \sum_{n=1}^{\alpha-1}(-1)^{n-1}\frac{\zeta(\alpha-n+1)}{k^n} + (-1)^{\alpha-1}\frac{H_k}{k^{\alpha}}\end{aligned}$$

Hence we have the general formula

$$\mathscr{C}(\alpha,k) = \sum_{n=1}^{\alpha-1}(-1)^{n-1}\frac{\zeta(\alpha-n+1)}{k^n} + (-1)^{\alpha-1}\frac{H_k}{k^\alpha}$$

Dividing by k^β and summing w.r.t to k

$$\sum_{k=1}^\infty \frac{\mathscr{C}(\alpha,k)}{k^\beta} = \sum_{n=1}^{\alpha-1}(-1)^{n-1}\zeta(\alpha-n+1)\zeta(\beta+n) + (-1)^{\alpha-1}\sum_{k=1}^\infty \frac{H_k}{k^{\alpha+\beta}}$$

Now we use the general formula

$$\sum_{n=1}^\infty \frac{H_n}{n^q} = \left(1+\frac{q}{2}\right)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(k+1)\zeta(q-k)$$

Hence we have

$$\sum_{n=1}^\infty \frac{H_n}{n^{\alpha+\beta}} = \left(1+\frac{\alpha+\beta}{2}\right)\zeta(\alpha+\beta+1) - \frac{1}{2}\sum_{k=1}^{\alpha+\beta-2}\zeta(k+1)\zeta(\alpha+\beta-k)$$

And the generalization is the following formula

$$\sum_{k=1}^\infty \frac{\mathscr{C}(\alpha,k)}{k^\beta} = \sum_{n=1}^{\alpha-1}(-1)^{n-1}\zeta(\alpha-n+1)\zeta(\beta+n) - \frac{1}{2}\sum_{n=1}^{\alpha+\beta-2}(-1)^{\alpha-1}\zeta(n+1)\zeta(\alpha+\beta-n) + (-1)^{\alpha-1}\left(1+\frac{\alpha+\beta}{2}\right)\zeta(\alpha+\beta+1)$$

We conclude by putting that

$$\sum_{k=1}^\infty\sum_{n=1}^\infty \frac{1}{k^q n^p(n+k)} = \sum_{k=1}^\infty \frac{\mathscr{C}(p,k)}{k^q}$$

13.10 Relation to polygamma

We can relate the generalized harmonic number to the polygamma function

$$H_k^{(p)} = \zeta(p) + (-1)^{p-1}\frac{\psi_{p-1}(k+1)}{(p-1)!}$$

proof

$$H_k^{(p)} = \sum_{n=1}^k \frac{1}{n^p} = \zeta(p) - \sum_{n=k+1}^\infty \frac{1}{n^p}$$

Now change the index in the sum $n = i+k+1$

$$H_k^{(p)} = \sum_{n=1}^k \frac{1}{n^p} = \zeta(p) - \sum_{i=0}^\infty \frac{1}{(i+k+1)^p}$$

We know that

$$(-1)^p \frac{\psi_{p-1}(k+1)}{(p-1)!} = \sum_{i=0}^{\infty} \frac{1}{(i+k+1)^p} \quad p \geq 1$$

Hence we have

$$H_k^{(p)} = \zeta(p) + (-1)^{p-1} \frac{\psi_{p-1}(k+1)}{(p-1)!}$$

We can use that to obtain a nice integral representation.

13.11 Integral representation for r=1

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} = \zeta(p)\zeta(q) + (-1)^p \frac{1}{(p-1)!} \int_0^1 \frac{\mathrm{Li}_q(x)\log(x)^{p-1}}{1-x} \, dx$$

proof

Note that

$$\psi_0(a+1) = \int_0^1 \frac{1-x^a}{1-x} \, dx$$

By differentiating with respect to a , p times we have

$$\psi_p(a+1) = \frac{\partial}{\partial a^p} \int_0^1 \frac{1-x^a}{1-x} \, dx$$

$$\psi_p(a+1) = -\int_0^1 \frac{x^a \log(x)^p}{1-x} \, dx$$

Let $a = k$

$$\psi_{p-1}(k+1) = -\int_0^1 \frac{x^k \log(x)^{p-1}}{1-x} \, dx$$

Use the relation to polygamma

$$H_k^{(p)} = \zeta(p) + (-1)^p \frac{1}{(p-1)!} \int_0^1 \frac{x^k \log(x)^{p-1}}{1-x} \, dx$$

Now divide by k^q and sum with respect to k

$$\sum_{k=1}^{\infty} \frac{H_k^{(p)}}{k^q} = \zeta(p)\zeta(q) + (-1)^p \frac{1}{(p-1)!} \int_0^1 \frac{\mathrm{Li}_q(x)\log(x)^{p-1}}{1-x} \, dx$$

13.12 Symmetric formula

$$\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q}+\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}=\zeta(p)\zeta(q)+\zeta(p+q)$$

proof

Take the leftmost series and swap the finite and infinite sums

$$\sum_{i=1}^{\infty}\sum_{k=i}^{\infty}\frac{1}{i^p}\frac{1}{k^q}=\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{i^p}\frac{1}{k^q}-\sum_{i=1}^{\infty}\frac{1}{i^p}\sum_{k=1}^{i-1}\frac{1}{k^q}$$

The second sum can be written as

$$\sum_{i=1}^{\infty}\frac{1}{i^p}\sum_{k=1}^{i-1}\frac{1}{k^q}=\sum_{i=1}^{\infty}\frac{1}{i^p}\left(\sum_{k=1}^{i}\frac{1}{k^q}-\frac{1}{i^p}\right)$$

By separating and changing the index we get

$$\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}-\zeta(p+q)$$

Hence we have

$$\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q}=\zeta(p)\zeta(q)-\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}+\zeta(p+q)$$

$$\sum_{k=1}^{\infty}\frac{H_k^{(p)}}{k^q}+\sum_{k=1}^{\infty}\frac{H_k^{(q)}}{k^p}=\zeta(p)\zeta(q)+\zeta(p+q)$$

For the special case $p=q=n$

$$\sum_{k=1}^{\infty}\frac{H_k^{(n)}}{k^n}=\frac{\zeta^2(n)+\zeta(2n)}{2}$$

13.13 Example

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^2}=\frac{11\zeta(5)}{2}-2\zeta(2)\zeta(3)$$

proof

Using the symmetric formula

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^2}=\zeta(2)\zeta(3)+\zeta(5)-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}$$

Using the integral formula on the second sum

$$\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^3}=\zeta(2)\zeta(3)+\int_0^1\frac{\mathrm{Li}_3(x)\log(x)}{1-x}\,dx$$

Using integration by parts on the integral

$$\int_0^1 \frac{\mathrm{Li}_3(x)\log(x)}{1-x}\,dx = -\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,dx$$

Let us think of solving

$$\int_0^1 \frac{\mathrm{Li}_2(x)\mathrm{Li}_2(1-x)}{x}\,dx$$

Using the duplication formula

$$\mathrm{Li}_2(1-x) = \zeta(2) - \mathrm{Li}_2(x) - \log(x)\log(1-x)$$

$$\int_0^1 \frac{\mathrm{Li}_2(x)(\zeta(2) - \mathrm{Li}_2(x) - \log(x)\log(1-x))}{x}\,dx$$

The first integral

$$\zeta(2)\int_0^1 \frac{\mathrm{Li}_2(x)}{x}\,dx = \zeta(2)\zeta(3)$$

The third integral

$$\int_0^1 \frac{\mathrm{Li}_2(x)\log(x)\log(1-x)}{x}\,dx = \frac{1}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

Finally we get

$$\int_0^1 \frac{\mathrm{Li}_3(x)\log(x)}{1-x}\,dx = \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx - \zeta(2)\zeta(3)$$

So

$$\sum_{k=1}^\infty \frac{H_k^{(2)}}{k^3} = \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

Hence we finally get that

$$\sum_{k=1}^\infty \frac{H_k^{(3)}}{k^2} = \zeta(2)\zeta(3) + \zeta(5) - \frac{3}{2}\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

Let us solve the integral

$$\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx$$

By series expansion

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(nk)^2}\int_0^1 x^{n+k-1}\,dx = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(nk)^2(n+k)}$$

By some manipulations we get

$$\sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{\infty}\frac{k}{n^2(n+k)} = \sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{k=1}^{\infty}\frac{1}{k^3}\sum_{n=1}^{\infty}\frac{1}{n(n+k)}$$

This can be simplified to conclude that

$$\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx = \zeta(2)\zeta(3) - \sum_{k=1}^{\infty}\frac{H_k}{k^4}$$

Now using that

$$\sum_{k=1}^{\infty}\frac{H_k}{k^4} = 3\zeta(5) - \zeta(2)\zeta(3)$$

Hence

$$\int_0^1 \frac{\mathrm{Li}_2^2(x)}{x}\,dx = 2\zeta(2)\zeta(3) - 3\zeta(5)$$

Finally we get

$$\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^2} = \zeta(2)\zeta(3) + \zeta(5) - \frac{3}{2}\left(2\zeta(2)\zeta(3) - 3\zeta(5)\right) = \frac{11\zeta(5)}{2} - 2\zeta(2)\zeta(3)$$

14 Sine Integral function

14.1 Definition

We define the following

$$\mathrm{Si}(z) = \int_0^z \frac{\sin(x)}{x}\,dx$$

A closely related function is the following

$$\mathrm{si}(z) = -\int_z^\infty \frac{\sin(x)}{x}\,dx$$

These functions are related through the equation

$$\mathrm{Si}(z) = \mathrm{si}(z) + \frac{\pi}{2}$$

A closely related function is the sinc function

$$\mathrm{sinc}(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin(x)}{x} & x \neq 0 \end{cases}$$

Using that we conclude

$$\frac{d}{dx}\mathrm{Si}(x) = \mathrm{sinc}(x)$$

For integration we have

$$\int \mathrm{Si}(x)\,dx = \cos(x) + x\,\mathrm{Si(x)} + C$$

14.2 Example

Show that

$$\int_0^\infty \sin(x)\mathrm{si}(x)\,dx = -\frac{\pi}{4}$$

proof

Using integration by parts we get

$$-\int_0^\infty \frac{\sin(x)\cos(x)}{x}dx = -\frac{1}{2}\int_0^\infty \frac{\sin(2x)}{x}\,dx$$

Let $2x = t$

$$-\frac{1}{2}\int_0^\infty \frac{\sin(t)}{t}dx = -\frac{\pi}{4}$$

14.3 Example

Prove

$$\int_0^\infty x^{\alpha-1}\,\mathrm{si}(x)\,dx = -\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)$$

proof

Using the integral representation

$$-\int_0^\infty x^{\alpha-1}\int_x^\infty \frac{\sin(t)}{t}\,dt\,dx$$

Let $xy = t$

$$-\int_0^\infty x^{\alpha-1}\int_1^\infty \frac{\sin(xy)}{y}\,dy\,dx$$

Switching the integrals we get

$$-\int_1^\infty \frac{1}{y}\int_0^\infty x^{\alpha-1}\,\sin(xy)\,dx\,dy$$

Now let $xy = t$

$$-\int_1^\infty \frac{1}{y^{\alpha+1}}\int_0^\infty t^{\alpha-1}\,\sin(t)\,dt\,dy$$

The Mellin transform of the sine function is defined as

$$\mathcal{M}_s(\sin(x)) = \int_0^\infty x^{s-1}\sin(x)\,dx = \Gamma(s)\sin\left(\frac{\pi s}{2}\right)$$

Hence we conclude that

$$-\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)\int_1^\infty \frac{1}{y^{\alpha+1}} = -\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)$$

14.4 Example

Show that

$$\int_0^\infty e^{-\alpha\, x}\,\mathrm{si}(x)\,dx = -\frac{\arctan(\alpha)}{\alpha}$$

proof

Use the integral representation

$$-\int_0^\infty e^{-\alpha\, x}\int_x^\infty \frac{\sin(t)}{t}\,dt\,dx$$

Let $xy = t$

$$-\int_0^\infty e^{-\alpha\, x}\int_1^\infty \frac{\sin(xy)}{y}\,dy\,dx$$

Switching the integrals

$$-\int_1^\infty \frac{1}{y}\int_0^\infty e^{-\alpha\, x}\,\sin(xy)\,dx\,dy$$

The inner integral is the laplace transform of the sine function

$$\mathcal{L}_s(\sin(at)) = \frac{a}{s^2+a^2}$$

Hence we conclude that

$$-\int_1^\infty \frac{1}{y^2+\alpha^2}\,dy = -\frac{\arctan(\alpha)}{\alpha}$$

14.5 Example

Prove the following

$$\int_0^\infty \mathrm{si}(x)\log(x)\,dx = \gamma + 1$$

proof We know that

$$\int_0^\infty x^{\alpha-1}\,\mathrm{si}(x)\,dx = -\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right)$$

Differentiate with respect to α

$$\int_0^\infty x^{\alpha-1}\,\mathrm{si}(x)\,\log(x)dx = \frac{\Gamma(\alpha)}{\alpha^2}\sin\left(\frac{\pi\alpha}{2}\right) - \frac{\Gamma(\alpha)\psi(\alpha)}{\alpha}\sin\left(\frac{\pi\alpha}{2}\right) - \frac{\pi\Gamma(\alpha)}{2\alpha}\cos\left(\frac{\pi\alpha}{2}\right)$$

Let $\alpha \to 1$

$$\int_0^\infty \mathrm{si}(x)\,\log(x)dx = 1 - \psi(1) = 1 - (-\gamma) = 1 + \gamma$$

14.6 Example

Find the integral

$$\int_0^\infty \mathrm{si}(x)\,\sin(px)\,dx$$

solution

Using integration by parts we get

$$\left[-\frac{\mathrm{si}(x)\cos(px)}{p}\right]_0^\infty + \frac{1}{p}\int_0^\infty \frac{\sin(x)}{x}\,\cos(px)\,dx$$

Taking the limits

$$\lim_{x\to 0} -\frac{\mathrm{si}(x)\cos(px)}{p} = \frac{\mathrm{si}(0)}{p} = \frac{\pi}{2p}$$

$$\lim_{x\to\infty} -\frac{\mathrm{si}(x)\cos(px)}{p} = 0$$

Hence we get

$$-\frac{\pi}{2p} + \frac{1}{p}\int_0^\infty \frac{\sin(x)}{x}\,\cos(px)\,dx$$

The integral

$$\int_0^\infty \frac{\sin(x)}{x}\,\cos(px)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x) - \sin((p-1)x)}{x}dx$$

Separate the integrals

$$I = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x)}{x}dx - \frac{1}{2}\int_0^\infty \frac{\sin((p-1)x)}{x}dx$$

If $p - 1 > 0$ we get

$$I = \frac{\pi}{4} - \frac{\pi}{4} = 0$$

If $p-1<0$

$$I = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x)}{x}dx + \frac{1}{2}\int_0^\infty \frac{\sin((1-p)x)}{x}dx = \frac{\pi}{2}$$

If $p=1$ we have

$$I = \frac{1}{2}\int_0^\infty \frac{\sin(2x)}{x}dx + 0 = \frac{\pi}{4}$$

Finally we get

$$\int_0^\infty \operatorname{si}(x)\sin(px)dx = \begin{cases} -\frac{\pi}{2p} & p>1 \\ -\frac{\pi}{4p} & p=1 \\ 0 & p<1 \end{cases}$$

14.7 Example

Prove that for $0<a<2$

$$\int_0^\infty \operatorname{si}^2(x)\,\cos(ax)\,dx = \frac{\pi}{2a}\log(a+1)$$

proof

Using integration by parts we get

$$\left[\frac{\operatorname{si}^2(x)\sin(ax)}{a}\right]_0^\infty - \frac{2}{a}\int_0^\infty \frac{\operatorname{si}(x)\sin(x)}{x}\,\sin(ax)\,dx$$

Taking the limits

$$\lim_{x\to 0}\frac{\operatorname{si}^2(x)\sin(ax)}{a} = 0$$

$$\lim_{x\to \infty}\frac{\operatorname{si}^2(x)\sin(ax)}{a} = 0$$

Let the integral

$$I(a) = \int_0^\infty \frac{\operatorname{si}(x)\sin(x)}{x}\,\sin(ax)\,dx$$

Differentiate with respect to a

$$I'(a) = \int_0^\infty \operatorname{si}(x)\sin(x)\,\cos(ax)\,dx$$

Now use the product to sum trigonometric rules

$$I'(a) = \frac{1}{2}\int_0^\infty \mathrm{si}(x)(\sin((a+1)x) - \sin((a-1)x))\,dx$$

From the previous exercise we have

$$\int_0^\infty \mathrm{si}(x)\sin((a+1)x)dx = \frac{-\pi}{4(a+1)} \;\; ; \, a > 0$$

$$\int_0^\infty \mathrm{si}(x)\sin((a+1)x)dx = 0 \;\; ; \, a < 2$$

Hence we conclude that for $0 < a < 2$

$$I'(a) = -\frac{\pi}{4(a+1)}$$

Integrate with respect to a

$$I(a) = -\frac{\pi}{4}\log(a+1) + C$$

Let $a \to 0$

$$I(0) = 0 + C \;\; \to \;\; C = 0$$

Hence we have

$$\int_0^\infty \frac{\mathrm{si}(x)\sin(x)}{x}\,\sin(ax)\,dx = -\frac{\pi}{4}\log(a+1)$$

Which implies that

$$\int_0^\infty \mathrm{si}^2(x)\,\cos(ax)\,dx = \frac{-2}{a}\left(-\frac{\pi}{4}\log(a+1)\right) = \frac{\pi}{2a}\log(a+1)$$

14.8 Example

Find the integral, for $a \neq 1$

$$\int_0^\infty \mathrm{si}(x)\cos(ax)\,dx$$

solution

Use integration by parts to obtain

$$\frac{1}{a}\int_0^\infty \frac{\sin(x)\sin(ax)}{x}\,dx$$

Let the integral

$$I(t) = \int_0^\infty e^{-tx}\frac{\sin(x)\sin(ax)}{x}\,dx$$

Differentiate with respect to t

$$I'(t) = -\int_0^\infty e^{-tx}\sin(x)\sin(ax)\,dx$$

Use product to sum rules

$$I'(t) = \frac{1}{2}\int_0^\infty e^{-tx}(\cos((a+1)x) - \cos((a-1)x))\,dx$$

Now we can use the Laplace transform

$$I'(t) = \frac{1}{2}\left(\frac{t}{t^2+(a+1)^2} - \frac{t}{t^2+(a-1)^2}\right)$$

Integrate with respect to t

$$I(t) = -\frac{1}{4}\log\left(\frac{t^2+(a+1)^2}{t^2+(a-1)^2}\right) + C$$

After verifying the constant goes to 0, we have

$$\int_0^\infty e^{-tx}\frac{\sin(x)\sin(ax)}{x}\,dx = -\frac{1}{4}\log\left(\frac{t^2+(a+1)^2}{t^2+(a-1)^2}\right)$$

Let $t \to 0$

$$\int_0^\infty \frac{\sin(x)\sin(ax)}{x}\,dx = -\frac{1}{4}\log\left(\frac{a+1}{a-1}\right)^2$$

We conclude that

$$\int_0^\infty \operatorname{si}(x)\cos(ax)\,dx = -\frac{1}{4a}\log\left(\frac{a+1}{a-1}\right)^2$$

15 Cosine Integral function

15.1 Definition

Define

$$\mathrm{ci}(x) = -\int_x^\infty \frac{\cos(t)}{t}\,dt$$

A related function is the following

$$\mathrm{Cin}(x) = \int_0^x \frac{1-\cos(t)}{t}\,dt$$

The derivative is

$$\frac{d}{dx}\mathrm{ci}(x) = \frac{\cos(x)}{x}$$

The integral

$$\int \mathrm{ci}(x)\,dx = x\mathrm{ci}(x) - \sin(x) + C$$

15.2 Relation to Euler constant

Prove that

$$\lim_{z\to\infty} \mathrm{Cin}(z) - \log z = \gamma$$

proof

Write the integral representation

$$\lim_{z\to\infty} \int_0^z \frac{1-\cos(t)}{t}dt - \log z$$

Can be written

$$\lim_{z\to\infty} \int_0^z \frac{1-\cos(t)}{t}dt - \int_0^z \frac{1}{1+t}dt = \int_0^\infty \frac{1}{t(1+t)} - \frac{\cos(t)}{t}dt$$

This is equivalent to

$$\lim_{s\to 0} \int_0^\infty \frac{t^{s-1}}{(1+t)} - t^{s-1}\cos(t)dt$$

The first integral

$$\int_0^\infty \frac{t^{s-1}}{(1+t)} = \Gamma(s)\Gamma(1-s)$$

The second integral

$$\int_0^\infty t^{s-1}\cos(t)dt = \Gamma(s)\cos(\pi s/2)$$

Hence, it reduces to evaluating the limit

$$\lim_{s\to 0}\Gamma(s)\Gamma(1-s) - \Gamma(s)\cos(\pi s/2)$$

Using $\Gamma(s+1) = s\Gamma(s)$

$$\lim_{s\to 0}\frac{\Gamma(1-s) - \cos(\pi s/2)}{s}$$

Use L'Hospital rule

$$\lim_{s\to 0} -\Gamma(1-s)\psi(1-s) + (\pi/2)\sin(\pi s/2) = -\psi(1) = \gamma$$

15.3 Example

Prove the following

$$\text{Cin}(x) = -\text{ci}(x) + \log(x) + \gamma$$

Start by

$$\text{Cin}(x) = \int_0^x \frac{1-\cos(t)}{t}\,dt$$

Rewrite as

$$\text{Cin}(x) = \int_0^\infty \frac{1-\cos(t)}{t}\,dt - \int_x^\infty \frac{1-\cos(t)}{t}\,dt$$

Which simplifies to

$$\text{Cin}(x) = \lim_{z\to\infty}\left[\int_0^z \frac{1-\cos(t)}{t}\,dt - \log(z)\right] - \text{ci}(x) + \log(x)$$

The limit goes to the Euler Maschorinit constant

$$\text{Cin}(x) = \gamma - \text{ci}(x) + \log(x)$$

15.4 Example

Find the integral

$$\int_0^\infty \operatorname{ci}(x)\,\cos(px)\,dx$$

solution

Using integration by parts we get

$$\left[\frac{\operatorname{ci}(x)\sin(px)}{p}\right]_0^\infty - \frac{1}{p}\int_0^\infty \frac{\cos(x)}{x}\,\sin(px)\,dx$$

Taking the limits

$$\lim_{x\to 0}\frac{\operatorname{ci}(x)\sin(px)}{p} = 0$$

$$\lim_{x\to \infty}\frac{\operatorname{ci}(x)\sin(px)}{p} = 0$$

Hence we get

$$-\frac{1}{p}\int_0^\infty \frac{\cos(x)}{x}\,\sin(px)\,dx$$

The integral

$$\int_0^\infty \frac{\cos(x)}{x}\,\sin(px)\,dx = \frac{1}{2}\int_0^\infty \frac{\sin((p-1)x)+\sin((p+1)x)}{x}dx$$

Separate the integrals

$$I = \frac{1}{2}\int_0^\infty \frac{\sin((p+1)x)}{x}dx + \frac{1}{2}\int_0^\infty \frac{\sin((p-1)x)}{x}dx$$

If $p-1>0$ we get

$$I = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

If $p-1<0$

$$I = \frac{\pi}{4} - \frac{\pi}{4} = 0$$

If $p=1$ we have

$$I = \frac{1}{2}\int_0^\infty \frac{\sin(2x)}{x} dx + 0 = \frac{\pi}{4}$$

Finally we get

$$\int_0^\infty \mathrm{ci}(x)\,\cos(px)\,dx = \begin{cases} -\frac{\pi}{2p} & p > 1 \\ -\frac{\pi}{4p} & p = 1 \\ 0 & p < 1 \end{cases}$$

15.5 Example

Find for $p > 1$

$$\int_0^\infty \mathrm{ci}(px)\mathrm{ci}(x)\,dx$$

solution

Let

$$I(p) = \int_0^\infty \mathrm{ci}(px)\mathrm{ci}(x)\,dx$$

Differentiate with respect to p

$$I'(p) = \frac{1}{p}\int_0^\infty \cos(px)\mathrm{ci}(x)\,dx$$

If $p > 1$ from the previous example we conclude that

$$I'(p) = \frac{1}{p}\left(\frac{-\pi}{2p}\right) = -\frac{\pi}{2p^2}$$

Integrate with respect to p

$$I(p) = \frac{\pi}{2p} + C$$

Take the limit $p \to \infty$, so $C = 0$.

15.6 Example

Prove that

$$\int_0^\infty x^{\alpha-1}\mathrm{ci}(x)\,dx = -\frac{\Gamma(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)$$

proof

Use the integral representation

$$-\int_0^\infty x^{\alpha-1} \int_x^\infty \frac{\cos(t)}{t}\,dt\,dx$$

Let $t = yx$

$$-\int_0^\infty x^{\alpha-1} \int_1^\infty \frac{\cos(yx)}{y}\,dy\,dx$$

Switch the integrals

$$-\int_1^\infty \frac{1}{y} \int_0^\infty x^{\alpha-1} \cos(yx)\,dx\,dy$$

Using the Mellin transform we get

$$-\Gamma(\alpha)\cos\left(\frac{\alpha\pi}{2}\right)\int_1^\infty \frac{1}{y^{1+\alpha}}\,dy = -\frac{\Gamma(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)$$

15.7 Example

Prove that

$$\int_0^\infty \mathrm{ci}(x)\log(x)\,dx = \frac{\pi}{2}$$

proof

From the previous example we know

$$\int_0^\infty x^{\alpha-1}\mathrm{ci}(x)\,dx = -\frac{\Gamma(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)$$

Differentiate with respect to α

$$\int_0^\infty x^{\alpha-1}\mathrm{ci}(x)\log(x)\,dx = \frac{\Gamma(\alpha)}{\alpha^2}\cos\left(\frac{\alpha\pi}{2}\right) - \frac{\Gamma(\alpha)\psi(\alpha)}{\alpha}\cos\left(\frac{\alpha\pi}{2}\right) + \frac{\pi}{2}\frac{\Gamma(\alpha)}{\alpha}\sin\left(\frac{\alpha\pi}{2}\right)$$

Take the limit $\alpha \to 1$

$$\int_0^\infty \mathrm{ci}(x)\log(x)\,dx = 0 - 0 + \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

15.8 Example

Show that

$$\int_0^\infty \mathrm{ci}(x) e^{-\alpha x}\, dx = -\frac{1}{\alpha} \log \sqrt{1+\alpha^2}$$

proof

Use the integral representation

$$-\int_0^\infty e^{-\alpha x} \int_1^\infty \frac{\cos(yx)}{y}\, dy\, dx$$

Switch the integrals

$$-\int_1^\infty \frac{1}{y} \int_0^\infty e^{-\alpha x} \cos(yx)\, dx\, dy$$

Use the Laplace transformation

$$-\int_1^\infty \frac{\alpha}{y(\alpha^2+y^2)}\, dy = -\frac{1}{2\alpha} \log(1+\alpha^2) = -\frac{1}{\alpha} \log \sqrt{1+\alpha^2}$$

16 Integrals involving Cosine and Sine Integrals

16.1 Example

Find the integral

$$\int_0^\infty \mathrm{si}(qx)\mathrm{ci}(x)\,dx$$

solution

Using the integral representation

$$-\int_0^\infty \mathrm{si}(qx)\int_1^\infty \frac{\cos(yx)}{y}\,dy\,dx$$

Switch the integrals

$$-\int_1^\infty \frac{1}{y}\int_0^\infty \mathrm{si}(qx)\cos(yx)\,dx\,dy$$

We also showed that

$$\int_0^\infty \mathrm{si}(x)\cos(ax)\,dx = -\frac{1}{2a}\log\left(\frac{a+1}{a-1}\right)$$

Let $a = y/q$

$$\int_0^\infty \mathrm{si}(x)\cos(yx/q)\,dx = -\frac{q}{2y}\log\left(\frac{y+q}{y-q}\right)$$

Let $x = tq$

$$\int_0^\infty \mathrm{si}(qt)\cos(yt)\,dx = -\frac{1}{2y}\log\left(\frac{y+q}{y-q}\right)$$

Substitue the value of the itnegral

$$\frac{1}{2}\int_1^\infty \frac{1}{y^2}\log\left(\frac{y+q}{y-q}\right)\,dy$$

We can prove that the anti-derivative

$$\left[\frac{\log(y)}{q} - \frac{1}{2q}\log(y^2-q^2) - \frac{1}{2y}\log\left(\frac{y+q}{y-q}\right)\right]_1^\infty$$

Which simplifies

$$\left[-\frac{1}{2q}\log\left(\frac{y^2-q^2}{y^2}\right) - \frac{1}{2y}\log\left(\frac{y+q}{y-q}\right)\right]_1^\infty$$

The limit $y \to \infty$

$$\lim_{y\to\infty} \frac{1}{2q}\log\left(\frac{y^2-q^2}{y^2}\right) + \frac{1}{2y}\log\left(\frac{y+q}{y-q}\right) = 0$$

The limit $y \to 1$

$$\frac{1}{2q}\log\left(1-q^2\right) + \frac{1}{2}\log\left(\frac{1+q}{1-q}\right)$$

Can be written as

$$\frac{1}{4q}\log\left(1-q^2\right)^2 + \frac{1}{4}\log\left(\frac{1+q}{1-q}\right)^2$$

16.2 Example

Prove that

$$\int_0^\infty \frac{\mathrm{ci}(\alpha x)}{x+\beta}\,dx = -\frac{1}{2}\left\{\mathrm{si}(\alpha\beta)^2 + \mathrm{ci}(\alpha\beta)^2\right\}$$

proof

Let the following

$$I(\alpha) = \int_0^\infty \frac{\mathrm{ci}(\alpha x)}{x+\beta}\,dx$$

Differentiate with respect to α

$$I'(\alpha) = \frac{1}{\alpha}\int_0^\infty \frac{\cos(\alpha x)}{x+\beta}\,dx$$

Let $x + \beta = t$

$$I'(\alpha) = \frac{1}{\alpha}\int_\beta^\infty \frac{\cos(\alpha(t-\beta))}{t}\,dt$$

Use trignometric rules

$$I'(\alpha) = \frac{1}{\alpha}\int_\beta^\infty \frac{\cos(\alpha t)\cos(\alpha\beta) + \sin(\alpha t)\sin(\alpha\beta)}{t}\,dt$$

Separate the integrals

$$I'(\alpha) = \frac{\cos(\alpha\beta)}{\alpha}\int_\beta^\infty \frac{\cos(\alpha t)}{t}\,dt + \frac{\sin(\alpha\beta)}{\alpha}\int_\beta^\infty \frac{\sin(\alpha t)}{t}\,dt$$

This simplifies to

$$I'(\alpha) = -\frac{\cos(\alpha\beta)}{\alpha}\mathrm{ci}(\alpha\beta) - \frac{\sin(\alpha\beta)}{\alpha}\mathrm{si}(\alpha\beta)$$

Integrate with respect to α

$$I(\alpha) = -\frac{1}{2}\left\{\mathrm{si}(\alpha\beta)^2 + \mathrm{ci}(\alpha\beta)^2\right\} + C$$

If $\alpha \to \infty$ we have $C = 0$.

17 Logarithm Integral function

17.1 Definition

Define =

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log(t)}$$

The derivative is

$$\frac{d}{dz}\mathrm{li}(z) = \frac{1}{\log(z)}$$

The integral

$$\int \mathrm{li}(z)\, dz = z\mathrm{li}(z) - \mathrm{Ei}(2\log z)$$

By using integration by parts

$$\int \mathrm{li}(z)\, dz = z\mathrm{li}(z) - \int_0^z \frac{x}{\log(x)} dx$$

In the integral let $-2\log(x) = t$

$$\int \mathrm{li}(z)\, dz = z\mathrm{li}(z) + \int_{-2\log(z)}^{\infty} \frac{e^{-t}}{t} dt = z\mathrm{li}(z) - \mathrm{Ei}(2\log z)$$

17.2 Example

Prove that

$$\int_0^1 \mathrm{li}(x)\, dx = -\log(2)$$

proof

Let the following

$$I(a) = \int_0^1 \int_0^x \frac{e^{-a\log(t)} dt}{\log(t)}\, dx$$

Differentiate with respect to a

$$I'(a) = -\int_0^1 \int_0^x t^{-a}\, dt\, dx$$

$$I'(a) = \frac{1}{a-1}\int_0^1 x^{1-a}\,dx$$

Which reduces to

$$I'(a) = \frac{1}{(a-1)(2-a)} = \frac{1}{2-a} - \frac{1}{1-a}$$

Integrate with respect to a

$$I(a) = \log\left(\frac{1-a}{2-a}\right) + C$$

Take the limit $a \to \infty$ we get $C = 0$

$$\int_0^1 \int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx = \log\left(\frac{1-a}{2-a}\right)$$

Let $a \to 0$

$$\int_0^1 \operatorname{li}(x)\,dx = \log\left(\frac{1}{2}\right) = -\log(2)$$

17.3 Find the integral

$$\int_0^1 x^{p-1}\operatorname{li}(x)\,dx$$

solution

Let the following

$$I(a) = \int_0^1 x^{p-1}\int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx$$

Differentiate with respect to a

$$I'(a) = -\int_0^1 x^{p-1}\int_0^x t^{-a}\,dt\,dx$$

$$I'(a) = \frac{1}{a-1}\int_0^1 x^{p-a}\,dx$$

Which reduces to

$$I'(a) = \frac{1}{(a-1)(p-a+1)} = \frac{1}{p}\left\{\frac{1}{p-a+1} - \frac{1}{1-a}\right\}$$

Integrate with respect to a

$$I(a) = \frac{1}{p}\log\left(\frac{1-a}{p-a+1}\right) + C$$

Take the limit $a \to \infty$ we get $C = 0$

$$\int_0^1 x^{p-1}\int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx = \frac{1}{p}\log\left(\frac{1-a}{p-a+1}\right)$$

Let $a \to 0$

$$\int_0^1 x^{p-1}\int_0^x \frac{e^{-a\log(t)}dt}{\log(t)}\,dx = \frac{1}{p}\log\left(\frac{1}{p+1}\right) = -\frac{1}{p}\log\left(p+1\right)$$

17.4 Find the integral

$$\int_0^1 \operatorname{li}\left(\frac{1}{x}\right)\sin(a\log(x))\,dx$$

proof

Let the following

$$I(b) = \int_0^1 \sin(a\log(x))\int_0^{\frac{1}{x}} \frac{e^{-b\log(t)}dt}{\log(t)}\,dx$$

Differentiate with respect to b

$$I'(b) = -\int_0^1 \sin(a\log(x))\int_0^{\frac{1}{x}} t^{-b}\,dt\,dx$$

$$I'(b) = \frac{1}{b-1}\int_0^1 x^{b-1}\sin(a\log(x))\,dx$$

Let $\log(x) = -t$

$$I'(b) = \frac{1}{1-b}\int_0^\infty e^{-tb}\sin(at)\,dt$$

Using the Laplace transform

$$I'(b) = \frac{a}{(b-1)(a^2+b^2)}$$

Integrate with respect to b

$$I(b) = \frac{a\log(a^2+b^2) - a\log(b-1)^2 + 2\arctan(b/a)}{2a^2+2} + C$$

Let $b \to \infty$

$$0 = \frac{\pi}{2(a^2+1)} + C$$

Hence we have

$$\int_0^1 \sin(a\log(x)) \int_0^{\frac{1}{x}} \frac{e^{-b\log(t)}dt}{\log(t)}\, dx = \frac{a\log(a^2+b^2) - a\log(b-1)^2 + 2\arctan(b/a)}{2a^2+2} - \frac{\pi}{2(a^2+1)}$$

Let $b \to 0$

$$\int_0^1 \sin(a\log(x))\mathrm{li}(x)\, dx = \frac{a\log(a^2)}{2a^2+2} - \frac{\pi}{2(a^2+1)} = \frac{1}{a^2+1}\left(a\log(a) - \frac{\pi}{2}\right)$$

17.5 Example

Find the integral

$$\int_0^1 \frac{\mathrm{li}(x)}{x} \log^{p-1}\left(\frac{1}{x}\right) dx$$

proof

Let the following

$$I(a) = \int_0^1 \frac{1}{x}\left[\int_0^x \frac{e^{-a\log(t)}}{\log(t)} dt\right] \log^{p-1}\left(\frac{1}{x}\right) dx$$

Differentiate with respect to a

$$I'(a) = -\int_0^1 \frac{1}{x}\left[\int_0^x t^{-a} dt\right] \log^{p-1}\left(\frac{1}{x}\right) dx$$

$$I'(a) = \frac{1}{a-1}\int_0^1 x^{-a} \log^{p-1}\left(\frac{1}{x}\right) dx$$

Let $-\log(x) = t$

$$I'(a) = \frac{1}{a-1}\int_0^\infty e^{-(1-a)t} t^{p-1}\, dt$$

$$I'(a) = -\frac{\Gamma(p)}{(1-a)(1-a)^p} = -\frac{\Gamma(p)}{(1-a)^{p+1}}$$

Integrate with respect to a

$$I(a) = -\frac{\Gamma(p)}{p(1-a)^p}$$

Let $a \to 0$, Hence

$$\int_0^1 \frac{\mathrm{li}(x)}{x} \log^{p-1}\left(\frac{1}{x}\right) dx = -\frac{\Gamma(p)}{p}$$

17.6 Example

Prove that

$$\int_1^\infty \mathrm{li}\left(\frac{1}{x}\right) \log^{p-1}(x)\, dx = -\frac{\pi}{\sin(\pi p)}\Gamma(p)$$

proof

Let the following

$$I(a) = \int_1^\infty \mathrm{li}\left(x^{-a}\right) \log^{p-1}(x)\, dx$$

Differentiate with respect to a

$$\frac{d}{da}\mathrm{li}\left(x^{-a}\right) = \frac{d}{da}\int_0^{x^{-a}} \frac{dt}{\log(t)} = \frac{x^{-a}}{a}$$

Hence we have

$$I'(a) = \frac{1}{a}\int_1^\infty x^{-a} \log^{p-1}(x)\, dx$$

Let $\log(x) = t$

$$I'(a) = \frac{1}{a}\int_0^\infty e^{-(a-1)t} t^{p-1}\, dt$$

Using the Laplace transform

$$I'(a) = \Gamma(p)\frac{1}{a(a-1)^p}$$

Take the integral

$$\int_1^\infty I'(a) da = \Gamma(p)\int_1^\infty \frac{1}{a(a-1)^p} da$$

The left hand-side

$$I(\infty) - I(1) = \Gamma(p)\int_1^\infty \frac{1}{a(a-1)^p} da$$

Now since $I(\infty) = 0$

$$I(1) = -\Gamma(p) \int_1^\infty \frac{1}{a(a-1)^p} da$$

Which implies that

$$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \log^{p-1}(x)\, dx = -\Gamma(p) \int_1^\infty \frac{1}{a(a-1)^p} da$$

Now let $t = a - 1$

$$\int_0^\infty \frac{t^{-p}}{t+1} dt$$

Using the beta integral $x + y = 1$ and $x - 1 = -p$ which implies that $x = 1 - p, y = p$

Hence we have

$$\int_0^\infty \frac{t^{-p}}{t+1} dt = \beta(p, 1-p) = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(\pi p)}$$

Finally we get

$$\int_1^\infty \operatorname{li}\left(\frac{1}{x}\right) \log^{p-1}(x)\, dx = -\frac{\pi}{\sin(\pi p)}\Gamma(p)$$

18 Clausen functions

18.1 Definition

Define

$$\mathrm{cl}_m(\theta) = \begin{cases} \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^m} & m \text{ is even} \\ \sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^m} & m \text{ is odd} \end{cases}$$

18.2 Duplication formula

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) - (-1)^m \mathrm{cl}_m(\pi - \theta))$$

proof

If m is even then

$$\mathrm{cli}_m(\pi - \theta) = \sum_{k=1}^{\infty} \frac{\sin(k\pi - k\theta)}{k^m} = -\sum_{k=1}^{\infty} (-1)^k \frac{\sin(k\theta)}{k^m}$$

This implies

$$\mathrm{cl}_2(\theta) + \mathrm{cli}_m(\pi - \theta) = \sum_{k=1}^{\infty} (-1)^k \frac{\sin(k\theta)}{k^m} + \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^m} = \frac{1}{2^{m-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\theta)}{k^m}$$

This implies that

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) - \mathrm{cl}_m(\pi - \theta))$$

If m is odd then

$$\mathrm{cli}_m(\pi - \theta) = \sum_{k=1}^{\infty} \frac{\cos(k\pi - k\theta)}{k^m} = \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\theta)}{k^m}$$

$$\sum_{k=1}^{\infty} \frac{\cos(k\theta)}{k^m} + \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\theta)}{k^m} = \frac{1}{2^{m-1}} \sum_{k=1}^{\infty} \frac{\sin(2k\theta)}{k^m}$$

Which implies that

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) + \mathrm{cl}_m(\pi - \theta))$$

Collecting the results we have

$$\mathrm{cl}_m(2\theta) = 2^{m-1}(\mathrm{cl}_m(\theta) - (-1)^m \mathrm{cl}_m(\pi - \theta))$$

18.3 Example

Find the integral, for m is even

$$\int_0^\pi \mathrm{cl}_m(\theta) d\theta$$

solution

Using the series representation

$$\int_0^\pi \sum_{k=1}^\infty \frac{\sin(k\theta)}{k^m} d\theta$$

Swap the integral and the series

$$\sum_{k=1}^\infty \frac{1}{k^m} \int_0^\pi \sin(k\theta) d\theta$$

The integral

$$\int_0^\pi \sin(k\theta) d\theta = -\left[\frac{1}{k}\cos(k\theta)\right]_0^\pi = \frac{-(-1)^k + 1}{k}$$

We get the summation

$$\sum_{k=1}^\infty \frac{-(-1)^k + 1}{k^{m+1}} = \zeta(m+1) + \eta(m+1)$$

Now use that

$$\eta(s) = (1 - 2^{1-s})\zeta(s)$$

$$\sum_{k=1}^\infty \frac{-(-1)^k + 1}{k^{m+1}} = \zeta(m+1) + (1 - 2^{-m})\zeta(m+1) = \zeta(m+1)(2 - 2^{-m})$$

18.4 Example

Find the integral for m is even

$$\int_0^\infty \mathrm{cl}_m(\theta) e^{-n\theta} d\theta$$

Using the series representation

$$\int_0^\infty \sum_{k=1}^\infty \frac{\sin(k\theta)}{k^m} e^{-n\theta} d\theta$$

Swap the integral and the series

$$\sum_{k=1}^{\infty} \frac{1}{k^m} \int_0^{\infty} \sin(k\theta) e^{-n\theta} \, d\theta$$

Using the Laplace transform we have

$$\sum_{k=1}^{\infty} \frac{1}{k^{m-1}(k^2+n^2)}$$

Add and subtract k^2 and divide by n^2

$$\frac{1}{n^2} \sum_{k=1}^{\infty} \frac{k^2+n^2-k^2}{k^{m-1}(k^2+n^2)}$$

Distribute the numerator

$$\frac{1}{n^2}\zeta(m-1) - \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{1}{k^{m-3}(k^2+n^2)}$$

Continue this approach to conclude that

$$\sum_{l=1}^{j} (-1)^{l-1} \frac{1}{n^{2l}} \zeta(m-(2l-1)) + \frac{(-1)^j}{n^{2j}} \sum_{k=1}^{\infty} \frac{1}{k^{m-(2j+1)}(k^2+n^2)}$$

Let $m - 2j - 1 = 1$ which implies that $j = m/2 - 1$

$$\sum_{l=1}^{m/2-1} (-1)^{l-1} \frac{1}{n^{2l}} \zeta(m-(2l-1)) + \frac{(-1)^{m/2-1}}{n^{m-2}} \sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)}$$

Now let us look at the sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \sum_{k=1}^{\infty} \frac{1}{2ink} \left\{ \frac{1}{k-in} - \frac{1}{k+in} \right\}$$

Which can be written as

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{1}{2n^2} \sum_{k=1}^{\infty} \frac{1}{k} \left\{ \frac{in}{k+in} + \frac{-in}{k-in} \right\}$$

According to the digamma function

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{1}{2n^2} \left\{ \gamma + \psi(1+in) + \psi(1-in) + \gamma \right\}$$

which simplifies to

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{\psi(1-in) + \psi(1+in) + 2\gamma}{2n^2}$$

Now we we can verify $\psi(1-in) = \overline{\psi(1+in)}$

Which suggests that

$$\psi(1+in) + \psi(1-in) = 2\Re\left\{\psi(1+in)\right\}$$

Hence we have the sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2+n^2)} = \frac{2\Re\left\{\psi(1+in)\right\} + 2\gamma}{2n^2} = \frac{\Re\left\{\psi(1+in)\right\} + \gamma)}{n^2}$$

This concludes to

$$\sum_{l=1}^{m/2-1} (-1)^{l-1} \frac{\zeta(m-(2l-1))}{n^{2l}} + (-1)^{m/2-1} \frac{\Re\left\{\psi(1+in)\right\} + \gamma}{n^m}$$

19 Clausen Integral function

19.1 Definiton

We define

$$\mathrm{cl}_2(x) = \sum_{k=1} \frac{\sin(kx)}{k^2}$$

19.2 Integral representation

$$\mathrm{cl}_2(\theta) = -\int_0^\theta \log\left[2\sin\left(\frac{\phi}{2}\right)\right] d\phi$$

proof

Start by the following

$$\mathrm{Li}_2(e^{i\theta}) = \sum_{k=1}^\infty \frac{e^{ik\theta}}{k^2} = \sum_{k=1}^\infty \frac{\cos(k\theta)}{k^2} + i\sum_{k=1}^\infty \frac{\sin(k\theta)}{k^2}$$

By the integral definition of the dilogarithm

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -\int_1^{e^{i\theta}} \frac{\log(1-x)}{x}\, dx$$

Let $x = e^{i\phi}$

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -i\int_0^\theta \log(1-e^{i\phi})d\phi$$

Let us look at the following

$$1 - e^{i\phi} = 1 - \cos(\phi) - i\sin(\phi) = 2\sin^2(\phi/2) - 2i\sin(\phi/2)\cos(\phi/2)$$

Which simplifies to

$$1 - e^{i\phi} = 2\sin(\phi/2)\left[\sin(\phi/2) - i\cos(\phi/2)\right] = 2\sin(\phi/2)e^{-(i/2)(\pi-\phi)}$$

Hence our integral

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -i\int_0^\theta \log\left[2\sin(\phi/2)e^{-(i/2)(\pi-\phi)}\right] d\phi$$

Use the complex logarithm properties

$$\mathrm{Li}_2(e^{i\theta}) - \zeta(2) = -i\int_0^\theta \log\left[2\sin(\phi/2)\right] d\phi + \frac{1}{4}(\pi-\theta)^2 - \frac{1}{4}\pi^2$$

By equating the imaginary parts we have our result.

We can see the special value

$$\operatorname{cl}_2\left(\frac{\pi}{2}\right) = \sum_{k=1}^{\infty} \frac{\sin(k\pi/2)}{k^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = G$$

Where G is the Catalan's constant.

19.3 Duplication formula

Prove the following

$$\operatorname{cl}_2(2\theta) = 2(\operatorname{cl}_2(\theta) - \operatorname{cl}_2(\pi - \theta))$$

proof

We provide a proof using the integral representation

$$\operatorname{cl}_2(2\theta) = -\int_0^{2\theta} \log\left[2\sin\left(\frac{t}{2}\right)\right] dt$$

Let $t = 2\phi$

$$-2\int_0^{\theta} \log\left[2\sin\left(\phi\right)\right] d\phi$$

Use the double angle identity

$$-2\int_0^{\theta} \log\left[4\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\right] d\phi$$

Separate the logarithms

$$-2\int_0^{\theta} \log\left[2\sin\left(\frac{\phi}{2}\right)\right] d\phi - 2\int_0^{\theta} \log\left[2\cos\left(\frac{\phi}{2}\right)\right] d\phi$$

We can verify that

$$\operatorname{cl}_2(\pi - \theta) = \int_0^{\theta} \log\left[2\cos\left(\frac{\phi}{2}\right)\right] d\phi$$

Hence

$$\operatorname{cl}_2(2\theta) = 2(\operatorname{cl}_2(\theta) - \operatorname{cl}_2(\pi - \theta))$$

Using that we get the value

$$\mathrm{cl}_2(3\pi) = 2\mathrm{cl}_2\left(\frac{3\pi}{2}\right) - 2\mathrm{cl}_2\left(-\frac{\pi}{2}\right)$$

Since $\mathrm{cl}_2(3\pi) = 0$

$$\mathrm{cl}_2\left(\frac{3\pi}{2}\right) = \mathrm{cl}_2\left(-\frac{\pi}{2}\right) = -\mathrm{cl}_2\left(\frac{\pi}{2}\right) = -G$$

19.4 Example

Prove that

$$\int_0^{2\pi} \mathrm{cl}_2(x)^2 dx = \frac{\pi^5}{90}$$

Using the series representation

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{(nk)^2} \int_0^{2\pi} \sin(kx)\sin(nx)dx$$

Consider the integral

$$\int_0^{2\pi} \sin(kx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \cos((k-n)x) - \cos((k+n)x)\,dx$$

We have two cases

If $n = k$ then

$$\frac{1}{2}\int_0^{2\pi} 1 - \cos(2nx)\,dx = \pi$$

If $n \neq k$

$$\frac{1}{2}\int_0^{2\pi} \cos((k-n)x) - \cos((k+n)x)\,dx = \frac{1}{2}\left[\frac{\sin((k-n)x)}{k-n} - \frac{\sin((k+n)x)}{k+n}\right]_0^{2\pi} = 0$$

Hence we have

$$\int_0^{2\pi} \sin(kx)\sin(nx)\,dx = \begin{cases} 0 & n \neq k \\ \pi & n = k \end{cases}$$

We can write the series as

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{(nk)^2} = \sum_{n\neq k}^{\infty} \frac{1}{(nk)^2} + \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Now since the integral $n \neq k$ goes to zero the result reduces to

$$\pi \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi\zeta(4) = \frac{\pi^5}{90}$$

19.5 Example

Prove that

$$\int_0^{\pi/2} x \log(\sin x)\, dx = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\log 2$$

proof

$$I = \int_0^{\frac{\pi}{2}} x \log(\sin x)\, dx = \int_0^{\frac{\pi}{2}} x \log(2 \sin x) - \frac{\pi^2}{8}\log(2)$$

The integral reduces to

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} x \log(2\sin x) dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \mathrm{Cl}_2(2\theta)\, d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n^2}\, d\theta \\
&= -\frac{1}{4}\sum_{n=1} \frac{(-1)^n}{n^3} + \frac{1}{4}\sum_{n=1} \frac{1}{n^3} \\
&= \frac{7}{16}\zeta(3)
\end{aligned}$$

Collecting that together we have

$$I = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\log(2)$$

19.6 Example

Prove that

$$\int_0^{\pi/4} x \cot(x)\, dx = -\frac{\pi}{8}\log(2) + G/2$$

proof

Start by integration by parts

$$\int_0^{\pi/4} x \cot(x)\, dx = \frac{\pi}{8}\log(2) - \int_0^{\pi/4} \log(\sin x)\, dx$$

In the integral

$$\int_0^{\pi/4} \log(\sin x)\, dx$$

Let $x \to t/2$

$$\frac{1}{2}\int_0^{\pi/2} \log(\sin t/2)\, dt$$

Which can be written as

$$\frac{1}{2}\int_0^{\pi/2} \log(2\sin t/2)\, dt - \frac{1}{2}\int_0^{\pi/2} \log(2)\, dt$$

Using the Clausen integral function we get

$$-\frac{1}{2}\mathrm{cl}_2(\pi/2) - \frac{\pi}{4}\log(2)$$

Note that $\mathrm{cl}_2(\pi/2) = G$

We deduce that

$$\int_0^{\pi/4} \log(\sin x)\, dx = -G/2 - \frac{\pi}{4}\log(2)$$

Collecting the results we have

$$\int_0^{\pi/4} x\cot(x)\, dx = \frac{\pi}{8}\log(2) + G/2 - \frac{\pi}{4}\log(2) = -\frac{\pi}{8}\log(2) + G/2$$

19.7 Second Integral representation

Prove that

$$\mathrm{cl}_2(\theta) = -\sin(\theta)\int_0^1 \frac{\log(x)}{x^2 - 2\cos(\theta)x + 1}dx$$

proof

Note that

$$x^2 - 2\cos(\theta)x + 1 = x^2 - (e^{i\theta} + e^{-i\theta})x + 1 = (x - e^{i\theta})(x - e^{-i\theta})$$

This implies

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{1}{e^{i\theta} - e^{-i\theta}}\left\{\frac{1}{x - e^{i\theta}} - \frac{1}{x - e^{-i\theta}}\right\}$$

Note that $e^{i\theta} - e^{-i\theta} = 2i\sin(\theta)$

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{1}{2i\sin(\theta)}\left\{\frac{1}{x - e^{i\theta}} - \frac{1}{x - e^{-i\theta}}\right\}$$

Now use the geometric series

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{-1}{2i\sin(\theta)}\left\{\sum_{k=0}^{\infty} x^k e^{-i(k+1)\theta} - \sum_{k=0}^{\infty} x^k e^{i(k+1)\theta}\right\}$$

$$\frac{1}{x^2 - 2\cos(\theta)x + 1} = \frac{1}{\sin(\theta)}\sum_{k=1}^{\infty} x^{k-1}\sin(k\theta)$$

That implies

$$-\sin(\theta)\int_0^1 \frac{\log(x)}{x^2 - 2\cos(\theta)x + 1}dx = -\sum_{k=1}^{\infty}\sin(k\theta)\int_0^1 x^{k-1}\log(x)\,dx = \mathrm{cl}_2(\theta)$$

19.8 Example

Find the value of

$$\mathrm{cl}_2\left(\frac{2\pi}{3}\right)$$

proof

Use the second integral representation

$$\mathrm{cl}_2(2\pi/3) = -\frac{\sqrt{3}}{2}\int_0^1 \frac{\log(x)}{x^2 + x + 1}dx$$

Use that

$$x^3 - 1 = (x-1)(x^2 + x + 1)$$

Hence

$$\mathrm{cl}_2(2\pi/3) = -\frac{\sqrt{3}}{2}\int_0^1 \frac{x-1}{x^3 - 1}\log(x)dx$$

Let $x^3 = t$

$$\mathrm{cl}_2(2\pi/3) = -\frac{1}{6\sqrt{3}}\int_0^1 \frac{t^{1/3-1}(t^{1/3} - 1)}{t - 1}\log(t)dx$$

Note that

$$\psi'(s) = \int_0^1 \frac{x^{s-1}}{1-x} \log(x)\, dx$$

We deduce that

$$\mathrm{cl}_2(2\pi/3) = -\frac{1}{6\sqrt{3}} \left(\psi'(2/3) - \psi'(1/3)\right)$$

20 Barnes G function

20.1 Definition

$$G(z+1) = (2\pi)^{z/2}\exp\left(-\frac{z+z^2(1+\gamma)}{2}\right)\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^n \exp\left(\frac{z^2}{2n}-z\right)\right\}$$

20.1.1 Functional equation

Prove that

$$G(z+1) = \Gamma(z)G(z)$$

proof

From the series representation we have

$$\frac{G(z+1)}{G(z)} = \sqrt{2\pi}\exp\left(-z-\gamma z+\frac{\gamma}{2}\right)\prod_{k=1}^{\infty}\left(\frac{k+z}{k+z-1}\right)^k \exp\left(\frac{2z-1-2k}{2k}\right).$$

This can be written as

$$\frac{G(z+1)}{G(z)} = z\sqrt{2\pi}\exp\left(-z+\frac{\gamma}{2}\right)\prod_{k=1}^{\infty}\left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right)\left(1+\frac{z}{k}\right)\frac{e^{-\gamma z}}{z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1} e^{\left(\frac{z}{k}\right)}$$

Use the definition of the gamma function

$$\frac{G(z+1)}{G(z)} = z\Gamma(z)\sqrt{2\pi}\exp\left(-z+\frac{\gamma}{2}\right)\prod_{k=1}^{\infty}\left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right)\left(1+\frac{z}{k}\right)$$

It suffices to prove that

$$z\sqrt{2\pi}\exp\left(-z+\frac{\gamma}{2}\right)\prod_{k=1}^{\infty}\left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right)\left(1+\frac{z}{k}\right) = 1$$

or

$$\prod_{k=1}^{\infty}\left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right)\left(1+\frac{z}{k}\right) = \frac{\exp\left(z-\frac{\gamma}{2}\right)}{z\sqrt{2\pi}}$$

Start by

$$\lim_{N\to\infty}\prod_{k=1}^{N}\left(\frac{k+z}{k+z-1}\right)^k \exp\left(-\frac{1+2k}{2k}\right)\left(1+\frac{z}{k}\right)$$

Notice

$$
\begin{aligned}
\prod_{k=1}^{N}\left(\frac{k+z}{k+z-1}\right)^{k}\left(1+\frac{z}{k}\right) &= \frac{\prod_{k=1}^{N}(k+z)^{k}\prod_{k=1}^{N}\left(1+\frac{z}{k}\right)}{\prod_{k=1}^{N}(k+z-1)^{k}} \\
&= \frac{\prod_{k=1}^{N}(k+z)^{k}\prod_{k=1}^{N}(k+z)}{zN!\prod_{k=1}^{N-1}(k+z)^{k+1}} \\
&= \frac{(N+z)^{N+1}\prod_{k=1}^{N-1}(k+z)^{k}\prod_{k=1}^{N-1}(k+z)}{zN!\prod_{k=1}^{N-1}(k+z)^{k+1}} \\
&= \frac{(N+z)^{N+1}\prod_{k=1}^{N-1}(k+z)^{k+1}}{zN!\prod_{k=1}^{N-1}(k+z)^{k+1}} \\
&= \frac{(N+z)^{N+1}}{zN!}
\end{aligned}
$$

The second product

$$
\prod_{k=1}^{N}\exp\left(-\frac{1+2k}{2k}\right) = \exp\left(-\sum_{k=1}^{N}\frac{1+2k}{2k}\right) = e^{-\frac{1}{2}H_N-N}
$$

Hence we have the following

$$
e^{-\frac{1}{2}H_N-N}\frac{(N+z)^{N+1}}{zN!}
$$

According to Stirling formula we have

$$
e^{-\frac{1}{2}H_N-N}\frac{(N+z)^{N+1}}{zN!} \sim e^{-\frac{1}{2}H_N-N}\frac{(N+z)^{N+1}}{z(N/e)^{N}} \times \frac{1}{\sqrt{2\pi N}}
$$

By some simplifications we have

$$
\frac{e^{-\frac{1}{2}(H_N-\log N)}}{z}\left(1+\frac{z}{N}\right)\times\left(1+\frac{z}{N}\right)^{N}\times\frac{1}{\sqrt{2\pi}} \sim \frac{\exp\left(-\frac{\gamma}{2}+z\right)}{z\sqrt{2\pi}}
$$

Where we used that

$$
\lim_{n\to\infty} H_n - \log(n) = \gamma
$$

and

$$
\lim_{n\to\infty}\left(1+\frac{z}{N}\right)^{N} = e^{z}
$$

20.2 Reflection formula

$$
\log\left\{\frac{G(1-z)}{G(1+z)}\right\} = z\log\left(\frac{\sin(\pi z)}{\pi}\right) + \frac{\mathrm{cl}_2(2\pi z)}{2\pi}
$$

proof

Start by the series expansion

$$\frac{G(1-z)}{G(1+z)} = \frac{(2\pi)^{-z/2}\exp\left(\frac{z-z^2(1+\gamma)}{2}\right)\prod_{n=1}^{\infty}\left\{\left(1-\frac{z}{n}\right)^n\exp\left(\frac{z^2}{2n}+z\right)\right\}}{(2\pi)^{z/2}\exp\left(-\frac{z+z^2(1+\gamma)}{2}\right)\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^n\exp\left(\frac{z^2}{2n}-z\right)\right\}}$$

This simplifies to

$$\frac{G(1-z)}{G(1+z)} = (2\pi)^{-z}e^z\prod_{n=1}^{\infty}\frac{(n-z)^n}{(n+z)^n}e^{2z}$$

Take the log of both sides

$$\log\left\{\frac{G(1-z)}{G(1+z)}\right\} = -z\log(2\pi)+z+\log\left\{\prod_{n=1}^{\infty}\frac{(n-z)^n}{(n+z)^n}e^{2z}\right\}$$

Let the following

$$f(z) = \log\left\{\prod_{n=1}^{\infty}\frac{(n-z)^n}{(n+z)^n}e^{2z}\right\} = \sum_{n=1}^{\infty} n\log(n-z) - n\log(n+z) + 2z$$

Differentiate with respect to z

$$f'(z) = \sum_{n=1}^{\infty}\frac{-n}{n-z} - \frac{n}{n+z} + 2 = \sum_{n=1}^{\infty}\frac{-n(n+z)-n(n-z)+2(n^2-z^2)}{n^2-z^2}$$

Hence we have

$$\sum_{n=1}^{\infty}\frac{-2z^2}{n^2-z^2}$$

Now we can use the following

$$z\pi\cot\pi z = 1 + \sum_{n=1}^{\infty}\frac{-2z^2}{n^2-z^2}$$

Hence we conclude that

$$f'(z) = z\pi\cot\pi z - 1$$

Integrate with respect to z

$$f(z) = \int_0^z x\pi\cot(\pi x)\,dx - z$$

Hence we have

$$\log\left\{\frac{G(1-z)}{G(1+z)}\right\} = -z\log(2\pi) + \int_0^z z\pi\cot(\pi x)\,dx$$

Now use integration by parts for the integral

$$\int_0^z x\pi\cot(\pi x)\,dx = z\log(\sin\pi z) - \int_0^z \log(\sin\pi x)dx \tag{1}$$

$$= z\log(2\sin\pi z) - \int_0^z \log(2\sin\pi x)dx \tag{2}$$

$$= z\log(2\sin\pi z) - \frac{1}{2\pi}\int_0^{2\pi z}\log\left(2\sin\frac{x}{2}\right)dx \tag{3}$$

$$= z\log(2\sin\pi z) + \frac{\mathrm{cl}_2(2\pi z)}{2\pi} \tag{4}$$

$$\tag{5}$$

That implies

$$\log\left\{\frac{G(1-z)}{G(1+z)}\right\} = z\log(2\sin\pi z) - z\log(2\pi) + \frac{\mathrm{cl}_2(2\pi z)}{2\pi} = z\log\left(\frac{\sin(\pi z)}{\pi}\right) + \frac{\mathrm{cl}_2(2\pi z)}{2\pi}$$

20.3 Values at positive integers

Prove that

$$G(n) = \prod_{k=1}^{n-1}\Gamma(k)$$

proof

It can be proved by induction. For $G(1) = 1$, suppose

$$G(n) = \prod_{k=1}^{n-1}\Gamma(k)$$

We want to show

$$G(n+1) = \prod_{k=1}^{n}\Gamma(k)$$

By the functional equation

$$G(n+1) = \Gamma(n)G(n) = \Gamma(n)\prod_{k=1}^{n-1}\Gamma(k) = \prod_{k=1}^{n}\Gamma(k)$$

20.4 Relation to Hyperfactorial function

We define the hyperfactorial function as

$$H(n) = \prod_{k=1}^{n} k^k$$

Prove for n is a positive integer

$$G(n+1) = \frac{(n!)^n}{H(n)}$$

proof

We can prove it by induction for $n = 0$ we have, $G(1) = 1$,

suppose that

$$G(n) = \frac{\Gamma(n)^{n-1}}{H(n-1)}$$

we want to show that

$$G(n+1) = \Gamma(n)G(n) = \Gamma(n)\frac{\Gamma(n)^{n-1}}{H(n-1)}$$

Notice that

$$H(n-1) = \prod^{n-1} k^k = \frac{\prod^n k^k}{n^n} = \frac{H(n)}{n^n}$$

We deduce that

$$G(n+1) = \Gamma(n)G(n) = \frac{\Gamma(n)^n \times n^n}{H(n)} = \frac{(n!)^n}{H(n)}$$

20.5 Loggamma integral

Prove that

$$\int_0^z \log\Gamma(x)dx = \frac{z}{2}\log(2\pi) + \frac{z(z-1)}{2} + z\log\Gamma(z) - \log G(z+1)$$

proof

Take the log to the series representation

$$\log G(z+1) = \frac{z}{2}\log(2\pi) - \frac{z + z^2(1+\gamma)}{2} + \sum_{n=1}^{\infty} n\log\left(1+\frac{z}{n}\right) + \frac{z^2}{2n} - z$$

Let the following

$$f(z) = \sum_{n=1}^{\infty} n \log\left(1 + \frac{z}{n}\right) + \frac{z^2}{2n} - z$$

Differentiate with respect to z

$$f'(z) = \sum_{n=1}^{\infty} \frac{n}{z+n} + \frac{z}{n} - 1 = \sum_{n=1}^{\infty} \frac{z^2}{n(n+z)}$$

Now use the following

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$$

which implies that

$$\sum_{n=1}^{\infty} \frac{z^2}{n(n+z)} = z\psi(z) + \gamma z + 1$$

Hence we have

$$f'(z) = z\psi(z) + \gamma z + 1$$

Integrate with respect to z

$$f(z) = \int_0^z x\psi(x)dx + \frac{\gamma z^2}{2} + z$$

which implies that

$$f(z) = z\log\Gamma(z) - \int_0^z \log\Gamma(x)dx + \frac{\gamma z^2}{2} + z$$

Hence we have

$$\log G(z+1) = \frac{z}{2}\log(2\pi) - \frac{z + z^2(1+\gamma)}{2} + z\log\Gamma(z) - \int_0^z \log\Gamma(x)dx + \frac{\gamma z^2}{2} + z$$

By some rearrangements we have

$$\int_0^z \log\Gamma(x)dx = \frac{z}{2}\log(2\pi) + \frac{z(z-1)}{2} + z\log\Gamma(z) - \log G(z+1)$$

20.6 Glaisher-Kinkelin constant

We define the Glaisher-Kinkelin constant as

$$A = \lim_{n\to\infty} \frac{H(n)}{n^{n^2/2+n/2+1/12}e^{-n^2/4}}$$

20.7 Relation to Glaisher-Kinkelin constant

Prove that

$$\lim_{n\to\infty} \frac{G(n+1)}{(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}} = \frac{e^{1/12}}{A}$$

proof

Use the relation to the hyperfactorial function

$$\lim_{n\to\infty} \frac{(n!)^n}{H(n)(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}}$$

Now use the Stirling approximation

$$(n!)^n \sim (2\pi)^{n/2}n^{n^2+n/2}e^{-n^2+1/12}$$

Hence we deduce that

$$\lim_{n\to\infty} \frac{(2\pi)^{n/2}n^{n^2+n/2}e^{-n^2+1/12}}{H(n)} \times \frac{1}{(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}}$$

By simplifications we have

$$e^{1/12} \lim_{n\to\infty} \frac{n^{n^2/2+n/2+1/12}e^{-n^2/4}}{H(n)} = \frac{e^{1/12}}{A}$$

20.8 Example

Prove that

$$\zeta'(2) = \frac{\pi^2}{6}\left(\log(2\pi) + \gamma - 12\log A\right)$$

We already proved that

$$\log\left[\frac{G(n+1)}{(2\pi)^{n/2}n^{n^2/2-1/12}e^{-3n^2/4}}\right] \sim \frac{1}{12} - \log A$$

Let the following

$$f(n) = \log\left[\frac{G(n+1)}{(2\pi)^{n/2} n^{n^2/2-1/12} e^{-3n^2/4}}\right]$$

Use the series representation of the Barnes functions

$$f(n) = \log\left[\frac{(2\pi)^{n/2}\exp\left(-\frac{n+n^2(1+\gamma)}{2}\right)\prod_{k=1}^{\infty}\left\{\left(1+\frac{n}{k}\right)^k \exp\left(\frac{n^2}{2k}-n\right)\right\}}{(2\pi)^{n/2} n^{n^2/2-1/12} e^{-3n^2/4}}\right]$$

Which reduces to

$$f(n) = -\frac{n+n^2(1+\gamma)}{2} + \sum_{k=1}^{\infty}\left\{k\log\left(1+\frac{n}{k}\right)+\frac{n^2}{2k}-n\right\} - \left(\frac{n^2}{2}-\frac{1}{12}\right)\log(n) + \frac{3n^2}{4}$$

Differentiate with respect to n

$$f'(n) = -\frac{1}{2} - n - \gamma n + n\psi(n) + \gamma n + 1 - n\log(n) - \frac{n}{2} + \frac{1}{12n} + \frac{3n}{2}$$

Note that we already showed that

$$\frac{d}{dn}\sum_{k=1}^{\infty}\left\{k\log\left(1+\frac{n}{k}\right)+\frac{n^2}{2k}-n\right\} = n\psi(n) + \gamma n + 1$$

By simplifications we have

$$f'(n) = n\psi(n) - n\log(n) + \frac{1}{12n} + \frac{1}{2}$$

Now use that

$$\psi(n) = \log(n) - \frac{1}{2n} - 2\int_0^{\infty}\frac{zdz}{(n^2+z^2)(e^{2\pi z}-1)}dz$$

Hence we deduce that

$$f'(n) = -2\int_0^{\infty}\frac{nzdz}{(n^2+z^2)(e^{2\pi z}-1)}dz + \frac{1}{12n}$$

Integrate with respect to n

$$f(n) = -\int_0^{\infty}\frac{z\log(n^2+z^2)}{(e^{2\pi z}-1)}dz + \frac{1}{12}\log(n) + C$$

Take the limit $n \to 0$

$$C = \lim_{n\to 0} f(n) - \frac{1}{12}\log(n) + \int_0^{\infty}\frac{z\log(z^2)}{(e^{2\pi z}-1)}dz$$

Hence we have the limit

$$\lim_{n\to 0} \frac{G(n+1)}{(2\pi)^{n/2} n^{n^2/2-1/12} e^{-3n^2/4}} - \frac{1}{12}\log(n) = \lim_{n\to 0} \log \frac{G(n+1)}{(2\pi)^{n/2} n^{n^2/2} e^{-3n^2/4}} = 0$$

Hence we see that

$$C = 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)}dz$$

Finally we have

$$f(n) = -\int_0^\infty \frac{z\log(n^2+z^2)}{(e^{2\pi z}-1)}dz + \frac{1}{12}\log(n) + 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)}dz$$

$$f(n) = -\int_0^\infty \frac{z\log\left(1+\frac{z^2}{n^2}\right)}{(e^{2\pi z}-1)}dz - \log(n^2)\int_0^\infty \frac{z}{(e^{2\pi z}-1)}dz + \frac{1}{12}\log(n) + 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)}dz$$

Also we have

$$\int_0^\infty \frac{z}{(e^{2\pi z}-1)}dz = \frac{1}{24}$$

That simplifies to

$$f(n) = -\int_0^\infty \frac{z\log\left(1+\frac{z^2}{n^2}\right)}{(e^{2\pi z}-1)}dz + 2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)}dz$$

Take the limit $n\to\infty$

$$2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)}dz = \frac{1}{12} - \log A$$

Now use that

$$\begin{aligned}
2\int_0^\infty \frac{z\log(z)}{(e^{2\pi z}-1)}dz &= 2\int_0^\infty \frac{z\log(z)}{e^{2\pi z}} \times \frac{1}{1-e^{-2\pi z}}dz \\
&= 2\sum_{n=0}^\infty \int_0^\infty e^{-2\pi z(n+1)} z\log(z)\,dz \\
&= \sum_{n=1}^\infty \frac{\psi(2)-\log(2\pi)+\log(n)}{2\pi^2 n^2} \\
&= \frac{(\psi(2)-\log(2\pi))\zeta(2)+\zeta'(2)}{2\pi^2}
\end{aligned}$$

Hence we conclude that

$$\zeta'(2) = (\log(2\pi)-\psi(2))\zeta(2) + 2\pi^2\left(\frac{1}{12}-\log A\right) = \zeta(2)(\log(2\pi)+\gamma-12\log A)$$

20.9 Example

Prove that

$$\zeta'(-1) = \frac{1}{12} - \log A$$

proof

Start by

$$\zeta(s) = \lim_{m\to\infty}\left(\sum_{k=1}^{m} k^{-s} - \frac{m^{1-s}}{1-s} - \frac{m^{-s}}{2} + \frac{sm^{-s-1}}{12}\right),\ \mathrm{Re}(s) > -3.$$

Differentiate with respect to s

$$\zeta'(s) = \lim_{m\to\infty}\left(-\sum_{k=1}^{m} k^{-s}\log(k) - \frac{m^{1-s}}{(1-s)^2} + \frac{m^{1-s}}{1-s}\log(m) + \frac{m^{-s}}{2}\log(m) + \frac{m^{-s-1}}{12} - \frac{m^{-s-1}}{12}\log(m)\right)$$

Now let $s \to -1$

$$\zeta'(-1) = \lim_{m\to\infty}\left(-\sum_{k=1}^{m} k\log(k) - \frac{m^2}{4} + \frac{m^2}{2}\log(m) + \frac{m}{2}\log(m) + \frac{1}{12} - \frac{1}{12}\log(m)\right)$$

Take the exponential of both sides

$$e^{\zeta'(-1)} = e^{1/12}\lim_{m\to\infty}\frac{m^{m^2/2+m/2-1/12}e^{-m^2/4}}{e^{\sum_{k=1}^{m} k\log(k)}} = e^{1/12}\lim_{m\to\infty}\frac{m^{m^2/2+m/2-1/12}e^{-m^2/4}}{H(m)} = \frac{e^{1/12}}{A}$$

We conclude that

$$\zeta'(-1) = \frac{1}{12} - \log A$$

20.10 Relation to Howrtiz zeta function

Prove that

$$\log G(z+1) - z\log\Gamma(z) = \zeta'(-1) - \zeta'(-1,z)$$

proof

Start by the following

$$\zeta(s,z) = \frac{z^{-s}}{2} + \frac{z^{1-s}}{s-1} + 2\int_0^\infty \frac{\sin(s\arctan(x/z))}{(z^2+x^2)^{s/2}(e^{2\pi x}-1)}\,dx$$

Take the derivative with respect to s and $s \to -1$

$$\zeta'(-1,z) = -\frac{z\log(z)}{2} + \frac{z^2\log(z)}{2} - \frac{z^2}{4} + \int_0^\infty \frac{x\log(x^2+z^2) + 2z\arctan(x/z)}{(e^{2\pi x}-1)}\,dx$$

Now use that

$$\psi(z) = \log(z) - \frac{1}{2z} - 2\int_0^\infty \frac{x}{(z^2+x^2)(e^{2\pi x}-1)}dx$$

Which implies that

$$\int_0^\infty \frac{2zx}{(z^2+x^2)(e^{2\pi x}-1)}dx = z\log(z) - \frac{1}{2} - z\psi(z)$$

By taking the integral

$$\int_0^\infty \frac{x\log(x^2+z^2) - x\log(x^2)}{(e^{2\pi x}-1)}dx = \int_0^z x\log(x)\,dx - \int_0^z x\psi(x)\,dx - \frac{z}{2}$$

Which simplifies to

$$\int_0^\infty \frac{x\log(x^2+z^2)}{(e^{2\pi x}-1)}dx = \zeta'(-1) - \frac{z^2}{4} + \frac{1}{2}z^2\log(z) - z\log\Gamma(z) + \int_0^z \log\Gamma(x)\,dx - \frac{z}{2}$$

Also we have

$$2\int_0^\infty \frac{x}{(x^2+z^2)(e^{2\pi x}-1)}dx = \log(z) - \frac{1}{2z} - \psi(z)$$

By integration we have

$$2\int_0^\infty \frac{\arctan(x/z)}{(e^{2\pi x}-1)}dx = z + \frac{\log(z)}{2} - z\log(z) + \log\Gamma(z) + C$$

Let $z \to 1$ to evaluate the constant

$$2\int_0^\infty \frac{\arctan(x/z)}{(e^{2\pi x}-1)}dx = z + \frac{\log(z)}{2} - z\log(z) + \log\Gamma(z) - \frac{1}{2}\log(2\pi)$$

Multiply by z

$$2\int_0^\infty \frac{z\arctan(x/z)}{(e^{2\pi x}-1)}dx = z^2 + \frac{z\log(z)}{2} - z^2\log(z) + z\log\Gamma(z) - \frac{z}{2}\log(2\pi)$$

Substitute both integrals in our formula

$$\int_0^z \log\Gamma(x)\,\mathrm{d}x = \frac{z}{2}\log(2\pi) + \frac{z(1-z)}{2} - \zeta^{'}(-1) + \zeta^{'}(-1,z)$$

We also showed that

$$\int_0^z \log\Gamma(x)\,\mathrm{d}x = \frac{z}{2}\log(2\pi) + \frac{z(1-z)}{2} + z\log\Gamma(z) - \log G(z+1)$$

By equating the equations we get our result.

20.11 Example

Prove that

$$G\left(\frac{1}{2}\right) = 2^{1/24}\pi^{-1/4}e^{1/8}A^{-3/2}$$

proof

We know that

$$\log G(z) + \log\Gamma(z) - z\log\Gamma(z) = \zeta'(-1) - \zeta'(-1,z)$$

Note that

$$\zeta\left(s,\frac{1}{2}\right) = (2^s-1)\zeta(s)$$

Which implies that

$$\zeta'\left(-1,\frac{1}{2}\right) = \frac{\log(2)}{2}\zeta(-1) - \frac{1}{2}\zeta'(-1)$$

Hence we have

$$\log G\left(\frac{1}{2}\right) + \frac{1}{2}\log\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\zeta'(-1) - \frac{\log(2)}{2}\zeta(-1)$$

Using that we have

$$G\left(\frac{1}{2}\right) = 2^{1/24}\pi^{-1/4}e^{\frac{3}{2}\zeta'(-1)}$$

Note that

$$\zeta(-1) = -\frac{1}{12}$$

This can be proved by the functional equation of the zeta function.

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